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AN EFFECTIVE LAGRANGIAN APPROACH FOR UNSTABLE PARTICLES

W. Beenakker[*)]

Physics Department, University of Durham, Durham DH1 3LE, England

F.A. Berends and **A.P. Chapovsky**[†)]

Instituut–Lorentz, University of Leiden, The Netherlands

ABSTRACT

We propose a novel procedure for handling processes that involve unstable intermediate particles. By using gauge-invariant effective Lagrangians it is possible to perform a gauge-invariant resummation of (arbitrary) self-energy effects. For instance, gauge-invariant tree-level amplitudes can be constructed with the decay widths of the unstable particles properly included in the propagators. In these tree-level amplitudes modified vertices are used, which contain extra gauge-restoring terms prescribed by the effective Lagrangians. We discuss the treatment of the phenomenologically important unstable particles, like the top-quark, the W- and Z-bosons, and the Higgs-boson, and derive the relevant modified Feynman rules explicitly.

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1 Introduction

Many of the interesting reactions at present-day and future collider experiments involve a multitude of unstable particles during the intermediate stages. In view of the high precision of the experiments, the proper treatment of these unstable particles has become a demanding exercise, since on-shell approximations are simply not good enough anymore. A proper treatment of unstable particles requires the resummation of the corresponding self-energies to all orders. In this way the singularities originating from the poles in the on-shell propagators are regularized by the imaginary parts contained in the self-energies, which are closely related to the decay widths of the unstable particles. The perturbative resummation itself involves a simple geometric series and is therefore easy to perform. However, this simple procedure harbours the serious risk of breaking gauge invariance. Gauge invariance is guaranteed order by order in perturbation theory. Unfortunately one takes into account only part of the higher-order terms by resumming the self-energies. This results in a mixing of different orders of perturbation theory and thereby jeopardizes gauge invariance, even if the self-energies themselves are extracted in a gauge-invarant way.

During recent years awareness has been raised regarding the seriousness of the problem. It was shown explicitly how these gauge-breaking effects, which are formally of higher order in the expansion parameter, can nevertheless have profound repercussions on physical observables [1, 2, 3, 4]. This applies in particular to kinematical situations that approach asymptotic limits, like space-like virtual photons close to the on-shell limit [1, 2, 3] or longitudinal gauge bosons at high energies [4]. These asymptotic regimes are characterized by strong gauge cancellations, which are governed by the Ward identities of the theory. Any small gauge-breaking effect can upset these intricate gauge cancellations and can therefore be amplified significantly.

A solution to the problem is provided by the so-called pole-scheme [5], which allows the gauge-invariant calculation of matrix elements in the presence of unstable particles. The pole-scheme amounts to a systematic expansion of the matrix elements around the complex poles in the unstable-particle propagators. This can be viewed as a prescription for performing an effective expansion in powers of Γ_i/M_i, where M_i and Γ_i stand for the masses and widths of the unstable particles. The residues in the pole expansion are physically observable and therefore gauge-invariant. In reactions with multiple unstable-particle resonances it is rather awkward to perform the complete pole-scheme expansion with all its subtleties in the treatment of the off-shell phase space [6]. Therefore one usually approximates the expansion by retaining only the terms with the highest degree of resonance. This approximation is called the leading-pole approximation. The accuracy of the approximation is typically $\mathcal{O}(\Gamma_i/M_i)$, making it a suitable tool for calculating radiative corrections, since in that case the errors are further suppressed by powers of the coupling constant [6]. The errors induced at the lowest-order level, however, are as large as the radiative corrections themselves. In view of the high precision of present-day collider experiments, this is not acceptable. Therefore either the lowest-order expansion has to be performed explicitly or an alternative gauge-invariant resummation method should be used.

A few years ago a dedicated method was developed for the gauge-invariant treatment of unstable gauge bosons [3, 4, 7, 8]. This so-called fermion-loop scheme exploits the fact that

the unstable gauge bosons decay exclusively into fermions (at lowest order). Based on this observation, it proved natural to resum the fermionic one-loop self-energies and include all other possible one-particle-irreducible fermionic one-loop corrections. This resummation of one-particle-irreducible fermionic one-loop corrections involves the closed subset of all $\mathcal{O}([N_c^f\alpha/\pi]^n)$ contributions (with N_c^f denoting the colour degeneracy of fermion f), which makes it manifestly gauge-invariant. Unfortunately this method does not work for particles that also have bosonic decay modes. Moreover, the inclusion of a full-fledged set of one-loop corrections in a lowest-order amplitude tends to overcomplicate things.

A more general and rapidly developing method is the so-called pinch technique (PT) [9]. This method can be viewed as an extension of the fermion-loop scheme to the bosonic sector. It amounts to a re-organization of the various one-loop Green's functions in terms of gauge-invariant off-shell building blocks, labelled by the kinematical characteristics of the terms that are included (e.g self-energy-like terms, vertex-like terms, etc.). These building blocks satisfy ghost-free tree-level Ward identities (like in the fermion-loop scheme) and can be combined into gauge-invariant amplitudes. All this is achieved by making explicit use of the full Ward identities of the theory. After having applied the pinching procedure, the one-loop PT self-energies can be resummed in the resonant amplitudes [10]. The gauge invariance of this resummation then follows from the tree-level-like Ward identities of the non-resummed (vertex-like, box-like, etc.) one-loop corrections (see Ref. [11] for a formal proof based on the background-field-method). The PT is therefore a suitable candidate for treating lowest-order reactions involving unstable particles, although the lowest-order amplitudes will be quite complicated in view of the full set of non-resummed one-loop corrections. The complexity of these mandatory non-resummed one-loop corrections grows strongly with the amount of final-state particles in the lowest-order reaction, just like in the fermion-loop scheme. The terminology 'lowest-order' refers to the fact that resonant amplitudes are dominated by the decay widths in the propagators, which are calculated in lowest order in the PT. In order to go beyond the lowest order, the imaginary parts of the two-loop self-energies are needed. Gauge invariance of the resummation procedure in turn requires the inclusion of the relevant imaginary parts of the other two-loop corrections. At the moment some first attempts are under way to extend the PT beyond the one-loop order [12]. However, there is still a long way to go. Developing a fundamental (non-diagrammatic) understanding of the PT might be the most important next step in this context.

So, the need remains for a novel, preferably non-diagrammatic method to solve the full set of Ward identities. Ideally speaking, such an alternative method should be applicable to arbitrary reactions, involving all possible unstable particles and an unspecified amount of stable external particles. At the same time the gauge-restoring terms should be kept to a minimum. In this paper we propose such a novel technique for tree-level processes. By using gauge-invariant non-local effective Lagrangians, it is possible to generate the self-energy effects in the propagators as well as the required gauge-restoring terms in the multi-particle interactions. These multi-particle interactions can be derived explicitly in a relatively concise form. Of course one should generate physically sensible self-energies, so that the tree-level calculations are phenomenologically meaningful.

The paper is organized as follows. In Sect. 2 we briefly discuss the essence of the gauge-invariance problem. The non-local effective-Lagrangian method for the resummation of self-energies is introduced in Sects. 3 and 4 within the framework of an unbroken non-abelian $SU(N)$ gauge theory with fermions. In Sect. 5 we apply the method to the treatment of unstable particles in the Standard Model and give some simple examples. In the appendices we list some useful non-local Feynman rules, thus demonstrating the application of the method.

2 The gauge-invariance problem: a simple example

In this section we show the origin of the gauge-invariance problem associated with the resummation of self-energies, which is a minimal requirement for treating unstable particles. To this end we consider the simple example of an unbroken non-abelian $SU(N)$ gauge theory with fermions and subsequently integrate out these fermions. This example also allows to make a direct link to the philosophy behind the fermion-loop scheme.

First we fix our notation and introduce some conventions, which will be used throughout this paper. The $SU(N)$ generators in the fundamental representation are denoted by T^a with $a = 1, \cdots, N^2-1$. They are normalized according to $\mathrm{Tr}\,(\mathrm{T}^a\mathrm{T}^b) = \delta^{ab}/2$ and obey the commutation relation $\left[\mathrm{T}^a, \mathrm{T}^b\right] = if^{abc}\,\mathrm{T}^c$. In the adjoint representation the generators F^a are given by $(\mathrm{F}^a)^{bc} = -if^{abc}$. The Lagrangian of the unbroken $SU(N)$ gauge theory with fermions can be written as

$$\mathcal{L}(x) = -\,\frac{1}{2}\,\mathrm{Tr}\left[\boldsymbol{F}_{\mu\nu}(x)\,\boldsymbol{F}^{\mu\nu}(x)\right] + \bar{\psi}(x)\,(i\,\not{D} - m)\,\psi(x), \tag{1}$$

with

$$\boldsymbol{F}_{\mu\nu} \equiv \mathrm{T}^a F^a_{\mu\nu} = \frac{i}{g}\,[D_\mu, D_\nu], \qquad D_\mu = \partial_\mu - ig\,\mathrm{T}^a A^a_\mu \equiv \partial_\mu - ig\boldsymbol{A}_\mu. \tag{2}$$

Here ψ is a fermionic N-plet in the fundamental representation of $SU(N)$ and A^a_μ are the (N^2-1) non-abelian $SU(N)$ gauge fields, which form a multiplet in the adjoint representation. The Lagrangian (1) is invariant under the $SU(N)$ gauge transformations

$$\psi(x) \to \psi'(x) = \mathrm{G}(x)\,\psi(x),$$

$$\boldsymbol{A}_\mu(x) \to \boldsymbol{A}'_\mu(x) = \mathrm{G}(x)\,\boldsymbol{A}_\mu(x)\,\mathrm{G}^{-1}(x) + \frac{i}{g}\,\mathrm{G}(x)\left[\partial_\mu \mathrm{G}^{-1}(x)\right], \tag{3}$$

with the $SU(N)$ group element defined as $\mathrm{G}(x) = \exp[ig\,\mathrm{T}^a\theta^a(x)]$. The covariant derivative D_μ and field strength $\boldsymbol{F}_{\mu\nu}$ both transform in the adjoint representation

$$D_\mu \to \mathrm{G}(x)\,D_\mu\,\mathrm{G}^{-1}(x), \qquad \boldsymbol{F}_{\mu\nu}(x) \to \mathrm{G}(x)\,\boldsymbol{F}_{\mu\nu}(x)\,\mathrm{G}^{-1}(x). \tag{4}$$

Since the Lagrangian is quadratic in the fermion fields, one can integrate them out exactly in the functional integral. The resulting effective action is then given by

$$i\,S_{\text{eff}}[J] = i\int d^4x\left\{-\,\frac{1}{2}\,\mathrm{Tr}\left[\boldsymbol{F}_{\mu\nu}(x)\,\boldsymbol{F}^{\mu\nu}(x)\right] + J^a_\mu(x)\,A^{a,\mu}(x)\right\} + \mathrm{Tr}\left[\ln(-\,\not{D} - i\,m)\right], \tag{5}$$

with $J^a_\mu(x)$ denoting the gauge-field sources. The trace on the right-hand side has to be taken in group, spinor, and coordinate space. As a next step one can expand the effective action in terms of the coupling constant

$$\begin{aligned}
\mathrm{Tr}\Big[\ln(-\not{D} - i\,m)\Big] &= \mathrm{Tr}\Big[\ln(-\not{\partial} - i\,m)\Big] + \mathrm{Tr}\Big[\ln\Big(1 + \frac{g}{i\,\not{\partial} - m}\,\not{\boldsymbol{A}}\Big)\Big] \\
&= \mathrm{Tr}\Big[\ln(-\not{\partial} - i\,m)\Big] + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\,\mathrm{Tr}\Big[\Big(\frac{g}{i\,\not{\partial} - m}\,\not{\boldsymbol{A}}\Big)^n\Big]. \qquad (6)
\end{aligned}$$

Note that the left-hand side of Eq. (6) is gauge-invariant as a result of the trace-log operation. In contrast, the separate terms of the expansion on the right-hand side are not gauge-invariant. This is due to the fact that, unlike in the abelian case, the non-abelian gauge transformation (3) mixes different powers of the gauge field A_μ in Eq. (6). Thus, if one truncates the series on the right-hand side of Eq. (6) one will in general break gauge invariance. From Eq. (6) it is also clear that the fermionic part of the effective action induces higher-order interactions between the gauge bosons.

What are these higher-order interactions? Let us consider the quadratic gauge-field contribution

$$-\,\frac{1}{2}\,\mathrm{Tr}\Big[\Big(\frac{g}{i\,\not{\partial} - m}\,\not{\boldsymbol{A}}\Big)^2\Big] = -\,\frac{1}{2}\int d^4x\,d^4y\,\mathrm{Tr}\Big[O(x,y)\,O(y,x)\Big], \qquad (7)$$

where

$$O(x,y) = g\,S_{\mathrm{F}}^{(0)}(x-y)\,\not{\boldsymbol{A}}(y) \qquad (8)$$

and $i\,S_{\mathrm{F}}^{(0)}(x-y) = <0\,|\,T(\psi(x)\,\bar{\psi}(y))\,|\,0>_{\mathrm{free}}$ is the free fermion propagator. The trace on the right-hand side of Eq. (7) has to be taken in group and spinor space. A quick glance at this quadratic gauge-field contribution reveals that it is just the one-loop self-energy of the gauge boson induced by a fermion loop. In the same way, the higher-order terms $\sim g^n A^n$ in Eq. (6) are just the fermion-loop contributions to the n-point gauge-boson vertices.

One can truncate the expansion in Eq. (6) at $n=2$, thus taking into account only the gauge-boson self-energy term and neglecting the fermion-loop contributions to the higher-point gauge-boson vertices. This is evidently the simplest procedure for performing the Dyson resummation of the fermion-loop self-energies. However, as was pointed out above, truncation of Eq. (6) at any finite order in g in general breaks gauge invariance. This leads to the important observation that, *although the resummed fermion-loop self-energies are gauge-independent by themselves, the resummation is nevertheless responsible for gauge-breaking effects in the higher-point gauge-boson interactions through its inherent mixed-order nature.* Another way of understanding this is provided by the gauge-boson Ward identities. Since the once-contracted n-point gauge-boson vertex can be expressed in terms of $(n-1)$-point vertices (see Sect. 4), it is clear that gauge invariance is violated if the self-energies are resummed without adding the necessary compensating terms to the higher-point vertices.

An alternative is to keep all the terms in Eq. (6). Then the matrix elements derived from the effective action will be gauge-invariant. Keeping all the terms means that we will have to take into account not only the fermion-loop self-energy in the propagator, but also all the

possible fermion-loop contributions to the higher-point gauge-boson vertices. This is exactly the prescription of the fermion-loop scheme (FLS) [3, 4, 7]. Although the FLS guarantees gauge invariance of the matrix elements, it has disadvantages as well. Its general applicability is limited to those situations where non-fermionic particles can effectively be discarded in the self-energies, as is for instance the case for Γ_W and Γ_Z at lowest order. Another disadvantage is that in the FLS one is forced to do the loop calculations, even when calculating lowest-order quantities. For example, the calculation of the tree-level matrix element for the process $e^+e^- \to 4f\gamma$ involves a four-point gauge-boson interaction, which has to be corrected by fermion loops in the FLS. This overcomplicates an otherwise lowest-order calculation.

It is clear that the FLS provides more than we actually need. It does not only provide gauge invariance for the Dyson resummed matrix elements at a given order in the coupling constant, but it also takes into account all fermion-loop corrections at that given order. In the vicinity of unstable particle resonances the imaginary parts of the fermion-loop self-energies are effectively enhanced by $\mathcal{O}(1/g^2)$ with respect to the other fermion-loop corrections. Therefore, what is really needed is only a minimal subset of the non-enhanced contributions such that gauge invariance is restored. In a sense one is looking for a minimal solution of a system of Ward identities. The FLS provides a solution, but this solution is far from minimal and is only practical for particles that decay exclusively into fermions. Since the decay of unstable particles is a physical phenomenon, it seems likely that there exists a simpler and more natural method for constructing a solution to a system of Ward identities, without an explicit reference to fermions. In the following sections we will try to indicate how such a solution can be constructed. Although the solution will be valid for arbitrary self-energies, in practical calculations one will take physically sensible self-energies just like the FLS does.

3 An effective-Lagrangian approach for fermions

In this section we propose a scheme that allows a gauge-invariant resummation of fermion self-energies in tree-level processes, without having to resort to a complete set of loop corrections. This scheme will form the basis for the treatment of unstable fermions in the SM, like the top-quark. The crucial ingredient in the scheme is the use of non-local effective Lagrangians. We start off by briefly repeating the non-local Lagrangian formalism of Ref. [13], where the concept of non-local Lagrangians was used for a completely different purpose.

The usual local $SU(N)$ gauge theory with fermions has a Lagrangian given by Eq. (1), which is invariant under the gauge transformations (3). For the gauging procedure of the non-local Lagrangians we will need one more ingredient, the *path-ordered exponential*, which is defined as

$$\begin{aligned} U(x,y) \;=\; U^\dagger(y,x) \;&=\; \mathrm{P}\exp\Big[-ig\int_x^y \boldsymbol{A}_\mu(\omega)\,d\omega^\mu\Big] \qquad (9)\\ &= \lim_{d\omega_i\to 0}\Big(1-ig\boldsymbol{A}_\mu(x)\,d\omega_1^\mu\Big)\Big(1-ig\boldsymbol{A}_\mu(x+d\omega_1)\,d\omega_2^\mu\Big)\ldots\Big(1-ig\boldsymbol{A}_\mu(y-d\omega_n)\,d\omega_n^\mu\Big). \end{aligned}$$

Here $d\omega^\mu$ is the element of integration along some path $\Omega(x,y)$ that connects the points x and

y. In principle we are free to choose this particular path. For reasons that will become clear from the discussion presented below, we will make our choice in such a way that

$$\delta^{(4)}(x-y)\int_x^y \boldsymbol{A}_\mu(\omega)\, d\omega^\mu = 0 \tag{10}$$

and

$$\partial_y^\mu \int_x^y f(\omega)\, d\omega^\nu = f(y)\, g^{\mu\nu}. \tag{11}$$

The first condition implies that the null path $\Omega(x,x)$ always has zero length, i.e. it does not involve a closed loop. The second condition fixes the properties of the path-ordered exponentials under differentiation. The so-defined path-ordered exponential transforms as

$$U(x,y) \to \mathrm{G}(x)\, U(x,y)\, \mathrm{G}^{-1}(y) \tag{12}$$

under the $SU(N)$ gauge transformations. It hence carries the gauge transformation from one space-time point to the other. With the help of the path-ordered exponential one can rewrite the local action corresponding to the fermionic part of the Lagrangian (1) according to

$$\mathcal{S}_{\mathrm{L}} = \int d^4x\, d^4y\, \bar\psi(x)\, \delta^{(4)}(x-y)\, (i\,\partial\!\!\!/_y - m)\, U(x,y)\, \psi(y). \tag{13}$$

Note that the action (13) and the one obtained from Eq. (1) are only equivalent because of the condition (10).

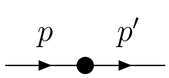
Using this gauging procedure we can now add a gauge-invariant non-local term to the Lagrangian:

$$\mathcal{L}_{\mathrm{NL}}(x,y) = \bar\psi(x)\, \Sigma_{\mathrm{NL}}(x-y)\, U(x,y)\, \psi(y). \tag{14}$$

It contains a non-local coefficient $\Sigma_{\mathrm{NL}}(x-y)$, which will play the role of a self-energy correction to the free fermion propagator. The argument of this coefficient is $x-y$ as a result of translational invariance. For calculational simplicity we will assume that $\Sigma_{\mathrm{NL}}(x-y)$ is mass-like, i.e. it is diagonal in spinor space. Consequently it has to be a function of the scalar invariant $(x-y)^2$. This condition is not essential for the method, but it will suit our purposes later on. With the new non-local term added to the Lagrangian, the total gauge-invariant fermionic action becomes

$$\begin{aligned}
\mathcal{S} &= \mathcal{S}_{\mathrm{L}} + \mathcal{S}_{\mathrm{NL}}, \\
\mathcal{S}_{\mathrm{L}} &= \int d^4x\, \bar\psi(x) \left[i\,\partial\!\!\!/ - m + g\, A\!\!\!/(x) \right] \psi(x), \\
\mathcal{S}_{\mathrm{NL}} &= \int d^4x\, d^4y\, \bar\psi(x)\, \Sigma_{\mathrm{NL}}(x-y)\, U(x,y)\, \psi(y).
\end{aligned} \tag{15}$$

As a next step we derive the relevant Feynman rules from the action $\mathcal{S}$. First we verify that the non-local term acts as a self-energy correction to the free fermion propagator:



$$: \; i\,\Sigma(y,z) = \left. \frac{i\,\delta^2\mathcal{S}}{\delta\psi(y)\,\delta\bar\psi(z)} \right|_{A=\psi=\bar\psi=0} = -\,(\partial\!\!\!/_z + i\,m)\,\delta^{(4)}(z-y) + i\,\Sigma_{\mathrm{NL}}(z-y). \tag{16}$$

Or, in momentum representation:

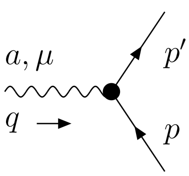
$$\tilde{\Sigma}(p,-p') = (2\pi)^4\,\delta^{(4)}(p-p')\,\Big[\not{p} - m + \tilde{\Sigma}_{\rm NL}(p^2)\Big], \tag{17}$$

where we performed the Fourier transforms

$$\begin{aligned} \Sigma_{\rm NL}(z-y) &= \frac{1}{(2\pi)^4}\int d^4l\, e^{-il\cdot(z-y)}\,\tilde{\Sigma}_{\rm NL}(l^2), \\ \Sigma(y,z) &= \frac{1}{(2\pi)^8}\int d^4p\, d^4p'\, e^{ip\cdot y}\, e^{-ip'\cdot z}\,\tilde{\Sigma}(p,-p'). \end{aligned} \tag{18}$$

By convention we use a tilde to indicate functions in the momentum representation. Upon inversion of Eq. (17), the non-local coefficient is indeed seen to act as a mass-like correction to the free fermion propagator.

Next we investigate how the gauge-boson–fermion–fermion vertex is modified by the non-local interaction. This vertex consists of two parts: a local and a non-local one. The local part is standard:



$$: \; ig\,\Gamma^{a,\,\mu}_{\rm L}(x,y,z) = \frac{i\,\delta^3\mathcal{S}_{\rm L}}{\delta A^a_\mu(x)\,\delta\psi(y)\,\delta\bar{\psi}(z)}\bigg|_{A=\psi=\bar{\psi}=0} = ig\,{\rm T}^a\,\delta^{(4)}(x-y)\,\delta^{(4)}(x-z)\,\gamma^\mu. \tag{19}$$

Or, in momentum representation:

$$ig\,\tilde{\Gamma}^{a,\,\mu}_{\rm L}(q,p,-p') = ig\,{\rm T}^a\,\gamma^\mu\,(2\pi)^4\,\delta^{(4)}(q+p-p'). \tag{20}$$

In order to calculate the non-local contribution to the vertex, it is convenient to take the Fourier transform of $\Sigma_{\rm NL}$, perform a Taylor expansion of $\tilde{\Sigma}_{\rm NL}(l^2)$, and finally perform integration by parts. Then $\mathcal{S}_{\rm NL}$ can be rewritten as

$$\begin{aligned} \mathcal{S}_{\rm NL} &= \sum_{n=0}^{\infty}\frac{1}{n!}\,\mathcal{S}^{(n)}_{\rm NL}\left(\frac{d}{dl^2}\right)^n\tilde{\Sigma}_{\rm NL}(l^2)\bigg|_{l^2=0} \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\,\mathcal{S}^{(n)}_{\rm NL}\,\tilde{\Sigma}^{(n)}_{\rm NL}(0), \\ \mathcal{S}^{(n)}_{\rm NL} &= \int d^4x\, d^4y\,\delta^{(4)}(x-y)\,\bar{\psi}(x)\,(-\partial^2_y)^n\,{\rm P}\exp\Bigg[-ig\int\limits_x^y \boldsymbol{A}_\mu(\omega)\,d\omega^\mu\Bigg]\psi(y). \end{aligned} \tag{21}$$

Of course, this expansion is not always allowed. We assume, however, that the non-local coefficient has analyticity properties that are similar to the ones expected for a normal self-energy function. As such we can perform the calculation in the regime where the above expansion is applicable and subsequently extend the range of validity by means of an analytical continuation. In fact, we will be able to present the non-local Feynman rules in such a way that the Ward identities are fulfilled irrespective of the precise form of the non-local coefficient. The non-local part of the gauge-boson–fermion–fermion vertex now reads

$$ig\,\Gamma^{a,\mu}_{\rm NL}(x,y,z) = \frac{i\,\delta^3\mathcal{S}_{\rm NL}}{\delta A^a_\mu(x)\,\delta\psi(y)\,\delta\bar{\psi}(z)}\bigg|_{A=\psi=\bar{\psi}=0} = ig\,{\rm T}^a\sum_{n=0}^{\infty}\frac{1}{n!}\,\tilde{\Sigma}^{(n)}_{\rm NL}(0)\,\mathcal{A}^\mu_n(z,y|x), \tag{22}$$

where $\mathcal{A}^{\mu}_{n}(z,y|x)$ is given by

$$\mathcal{A}^{\mu}_{n}(z,y|x) = -\,i\int d^4\tau\,\delta^{(4)}(z-\tau)\,(-\partial^2_\tau)^n\,\delta^{(4)}(\tau-y)\int\limits_{z}^{\tau}\delta^{(4)}(\omega-x)\,d\omega^{\mu}. \tag{23}$$

The corresponding Fourier transform can be simplified by eliminating some δ-function integrations and performing integration by parts:

$$\tilde{\mathcal{A}}^{\mu}_{n}(-p',p|q) = -\,i\int d^4\xi\,d^4\tau\,\delta^{(4)}(\xi-\tau)\,e^{ip'\cdot\xi}\,(-\partial^2_\tau)^n\,e^{-ip\cdot\tau}\int\limits_{\xi}^{\tau}e^{-iq\cdot\omega}\,d\omega^{\mu}. \tag{24}$$

By working out one of the $(-\partial^2_\tau)$ operators with the help of Eq. (11), one can derive the recursion relation

$$\tilde{\mathcal{A}}^{\mu}_{n}(-p',p|q) = p^2\tilde{\mathcal{A}}^{\mu}_{n-1}(-p',p|q) + (2p+q)^{\mu}\,(p+q)^{2n-2}\,(2\pi)^4\,\delta^{(4)}(q+p-p'). \tag{25}$$

From the base of the recursion, $\tilde{\mathcal{A}}^{\mu}_{0}(-p',p|q)=0$, it is clear that all terms in the series will be proportional to $(2p+q)^{\mu}\,(2\pi)^4\,\delta^{(4)}(q+p-p')$. The solution of the recursion relation can be found in App. A:

$$\tilde{\mathcal{A}}^{\mu}_{n}(-p',p|q) = (2p+q)^{\mu}\,\frac{(q+p)^{2n}-p^{2n}}{(q+p)^2-p^2}\,(2\pi)^4\,\delta^{(4)}(q+p-p'). \tag{26}$$

Substituting this into the definition of the gauge-boson–fermion–fermion vertex (22), one obtains for the non-local contribution in momentum representation

$$\begin{aligned} ig\,\tilde{\Gamma}^{a,\mu}_{\mathrm{NL}}(q,p,-p') &= ig\,\mathrm{T}^a\,(p+p')^{\mu}\sum_{n=0}^{\infty}\frac{1}{n!}\,\tilde{\Sigma}^{(n)}_{\mathrm{NL}}(0)\,\frac{p'^{\,2n}-p^{2n}}{p'^{\,2}-p^2}\,(2\pi)^4\,\delta^{(4)}(q+p-p') \\ &= ig\,\mathrm{T}^a\,\frac{(p+p')^{\mu}}{p'^{\,2}-p^2}\left[\tilde{\Sigma}_{\mathrm{NL}}(p'^{\,2})-\tilde{\Sigma}_{\mathrm{NL}}(p^2)\right](2\pi)^4\,\delta^{(4)}(q+p-p'). \end{aligned} \tag{27}$$

This expression exhibits the proper infrared behaviour,

$$\tilde{\Gamma}^{a,\mu}_{\mathrm{NL}}(q,p,-p') \xrightarrow{q\to 0} \mathrm{T}^a\,(2\pi)^4\,\delta^{(4)}(q+p-p')\,\frac{\partial\,\tilde{\Sigma}_{\mathrm{NL}}(p^2)}{\partial p_{\mu}}, \tag{28}$$

required for guaranteeing the usual eikonal factorization in the infrared limit.

It is easy to verify that the so-obtained full gauge-boson–fermion–fermion vertex satisfies the Ward identity for dressed fermion propagators:

$$q_{\mu}\left[\tilde{\Gamma}^{a,\mu}_{\mathrm{L}}(q,p,-p')+\tilde{\Gamma}^{a,\mu}_{\mathrm{NL}}(q,p,-p')\right] = (2\pi)^4\,\delta^{(4)}(q+p-p')\,\mathrm{T}^a\left[\tilde{S}^{-1}_F(p')-\tilde{S}^{-1}_F(p)\right], \tag{29}$$

with $\tilde{S}^{-1}_F(p) = \not{p} - m + \tilde{\Sigma}_{\mathrm{NL}}(p^2)$. From this we can conclude that the described non-local approach allows a gauge-invariant resummation of fermion self-energies, while at the same time reducing the complexity of the necessary gauge-restoring higher-point interactions to a

minimum. The general higher-point interaction between two fermions and k gauge bosons reads

$$
\begin{aligned}
ig^k\,\Gamma_{\rm NL}^{a_1..a_k,\,\mu_1..\mu_k}(x_1,..,x_k,y,z) &= \left.\frac{i\,\delta^{k+2}\mathcal{S}_{\rm NL}}{\delta A_{\mu_1}^{a_1}(x_1)\,..\,\delta A_{\mu_k}^{a_k}(x_k)\,\delta\psi(y)\,\delta\bar{\psi}(z)}\right|_{A=\psi=\bar{\psi}=0} \\
&= ig^k\sum_{n=0}^{\infty}\frac{1}{n!}\,\tilde{\Sigma}_{\rm NL}^{(n)}(0)\sum_{\rm perm}\mathcal{A}_n^{\nu_1..\nu_k}(z,y|y_1,..,y_k)\,{\rm T}^{b_1}\cdots{\rm T}^{b_k}. \qquad (30)
\end{aligned}
$$

The latter sum involves all possible permutations $\{(b_1,\nu_1,y_1),\ldots,(b_k,\nu_k,y_k)\}$ of the basic set $\{(a_1,\mu_1,x_1),\ldots,(a_k,\mu_k,x_k)\}$. The Fourier transform of the path-ordered tensor-function $\mathcal{A}_n^{\mu_1..\mu_k}(z,y|x_1,..,x_k)$ is given by

$$
\tilde{\mathcal{A}}_n^{\mu_1..\mu_k}(-p',p|q_1,..,q_k) = (-i)^k\int d^4\xi\,d^4\tau\,\delta^{(4)}(\xi-\tau)\,e^{ip'\cdot\xi}(-\partial_\tau^2)^n\,e^{-ip\cdot\tau}\prod_{j=1}^{k}\int_{\omega_{j-1}}^{\tau}d\omega_j^{\mu_j}\,e^{-iq_j\cdot\omega_j}, \quad (31)
$$

with $\omega_0=\xi$. In App. A we discuss briefly the recursion relations corresponding to these tensor-functions and present the solution for general k. Based on the simple 'Ward identities' for the tensor-functions given in App. A, it is easy to verify that the general interaction (30) obeys the Ward identity

$$
\begin{aligned}
q_{r,\,\mu_r}\,\tilde{\Gamma}_{\rm NL}^{a_1..a_k,\,\mu_1..\mu_k}(q_1,..,q_k,p,-p') &= \tilde{\Gamma}_{\rm NL}^{a_1..<a_r>..a_k,\,\mu_1..<\mu_r>..\mu_k}(q_1,..,<q_r>,..,q_k,p+q_r,-p')\,{\rm T}^{a_r} \\
&\quad - {\rm T}^{a_r}\,\tilde{\Gamma}_{\rm NL}^{a_1..<a_r>..a_k,\,\mu_1..<\mu_r>..\mu_k}(q_1,..,<q_r>,..,q_k,p,-p'+q_r) \\
&\quad - \sum_{j\neq r}({\rm F}^{a_r})^{a_j d}\left[\tilde{\Gamma}_{\rm NL}^{a_1..<a_r>..a_k,\,\mu_1..<\mu_r>..\mu_k}(q_1,..,<q_r>,..,q_k,p,-p')\right]_{\substack{a_j\to d\\ q_j\to q_j+q_r}}. \qquad (32)
\end{aligned}
$$

Here $<i_r>$ indicates that the index i_r has been removed. The substitutions in the last term should be applied to the expression inside the brackets only. The $SU(N)$ generator in the adjoint representation ${\rm F}^{a_r}$ has been defined in the previous section.

From all this we can conclude that the above-described procedure allows the gauge-invariant resummation of fermion self-energies in the context of a $SU(N)$ symmetric theory. At the same time the compensating terms in the higher-point interactions are kept to a minimum. It should be noted, however, that this procedure is not sufficient for a gauge-invariant description of unstable fermions in the Standard Model, since the symmetry is explicitly broken in that case. We will come back to this point in Sect. 5.

4 An effective-Lagrangian approach for gauge bosons

The next step is to extend the non-local method to the gauge-boson sector. We remind the reader that the non-local Lagrangian should allow the Dyson resummation of the gauge-boson self-energies, in order to make the link to unstable gauge bosons later on, and it should preserve

gauge invariance with a minimum of additional higher-point interactions. The starting-point of the non-local effective Lagrangian should therefore be a bilinear gauge-boson interaction. In the light of the discussion presented in Sect. 2, the main idea is to rearrange the series on the right-hand side of Eq. (6) in such a way that each term becomes gauge-invariant by itself. Subsequent truncation of the series at a given term is then allowed.

Since the gauge bosons transform in the adjoint representation ($\boldsymbol{F} \to \boldsymbol{F}' = \mathrm{G}\boldsymbol{F}\mathrm{G}^{-1}$), the non-local action for gauge bosons differs from the one for fermions in the way the path-ordered exponentials occur. For an $SU(N)$ Yang–Mills theory it takes the form

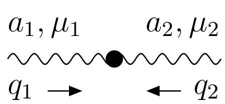
$$\mathcal{S}_{\mathrm{NL}} = -\frac{1}{2}\int d^4x\, d^4y\, \Sigma_{\mathrm{NL}}(x-y)\,\mathrm{Tr}\Big[U(y,x)\,\boldsymbol{F}_{\mu\nu}(x)\,U(x,y)\,\boldsymbol{F}^{\mu\nu}(y)\Big], \tag{33}$$

or, expanding the non-local coefficient Σ_{NL} in terms of derivatives,

$$\begin{aligned}
\mathcal{S}_{\mathrm{NL}} &= \sum_{n=0}^{\infty}\frac{1}{n!}\,\tilde{\Sigma}^{(n)}_{\mathrm{NL}}(0)\,\mathcal{S}^{(n)}_{\mathrm{NL}},\\
\mathcal{S}^{(n)}_{\mathrm{NL}} &= -\frac{1}{2}\int d^4x\, d^4y\, \delta^{(4)}(x-y)\,(-\partial_y^2)^n\,\mathrm{Tr}\Big[U(y,x)\,\boldsymbol{F}_{\mu\nu}(x)\,U(x,y)\,\boldsymbol{F}^{\mu\nu}(y)\Big].
\end{aligned} \tag{34}$$

As required, the action contains bilinear gauge-boson interactions. The induced infinite tower of higher-point gauge-boson interactions, which are also of progressively higher order in the coupling constant g, is needed for restoring gauge invariance.

Let us derive the relevant Feynman rules, starting with the two-point function



$$: \; i\,\Sigma^{a_1a_2,\,\mu_1\mu_2}(x_1,x_2) = \left.\frac{i\,\delta^2(\mathcal{S}_{\mathrm{L}}+\mathcal{S}_{\mathrm{NL}})}{\delta A^{a_1}_{\mu_1}(x_1)\,\delta A^{a_2}_{\mu_2}(x_2)}\right|_{A=0}, \tag{35}$$

where the local action $\mathcal{S}_{\mathrm{L}}$ follows from the gauge-boson term in Eq. (1). The Fourier transform of this two-point function can be calculated in a straightforward way, since the path-ordered exponentials are effectively unity. The result reads

$$i\,\tilde{\Sigma}^{a_1a_2,\,\mu_1\mu_2}(q_1,q_2) = i\,\delta^{a_1a_2}\Big(q_1^{\mu}q_1^{\nu} - q_1^2 g^{\mu\nu}\Big)\Big[1+\tilde{\Sigma}_{\mathrm{NL}}(q_1^2)\Big]\,(2\pi)^4\,\delta^{(4)}(q_1+q_2). \tag{36}$$

Note that this two-point interaction is transverse, as it should be for an unbroken theory. The non-local coefficient acts as a (dimensionless) correction to the transverse free gauge-boson propagator. So, exactly what is needed for the Dyson resummation of the gauge-boson self-energies.

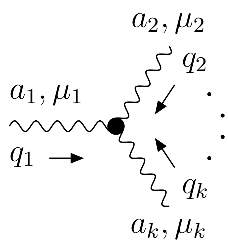
The general non-local interaction between k gauge bosons consists of four distinct contributions, with either two, three or four gauge fields supplied by the field strengths in Eq. (33). In order to simplify the derivation of the Feynman rules, it is convenient to write Eq. (33) in the adjoint representation rather than the fundamental representation:

$$\mathcal{S}_{\mathrm{NL}} = -\frac{1}{4}\int d^4x\, d^4y\, \Sigma_{\mathrm{NL}}(x-y)\, F^a_{\mu\nu}(x)\, U^{ab}(x,y)\, F^{\mu\nu,\,b}(y), \tag{37}$$

with

$$U^{ab}(x,y) = \mathrm{P}\exp\Big[-ig\int_x^y \mathrm{F}^c A^c_\mu(\omega)\,d\omega^\mu\Big]^{ab}. \tag{38}$$

For the general interaction between k gauge bosons we now obtain



$$: \quad ig^{k-2}\,\Gamma^{a_1..a_k,\,\mu_1..\mu_k}_{\mathrm{NL}}(x_1,..,x_k) \;=\; \frac{i\,\delta^k \mathcal{S}_{\mathrm{NL}}}{\delta A^{a_1}_{\mu_1}(x_1)\,..\,\delta A^{a_k}_{\mu_k}(x_k)}\bigg|_{A=0}$$

$$= \; ig^{k-2}\sum_{n=0}^{\infty}\frac{1}{n!}\,\tilde{\Sigma}^{(n)}_{\mathrm{NL}}(0)\sum_{\mathrm{perm}}(\mathrm{F}^{b_3}\cdots\mathrm{F}^{b_k})^{b_1 b_2}\,V^{\nu_1..\nu_k}_{\mathrm{NL},\,n}(y_1,..,y_k), \tag{39}$$

where the sum involves all possible permutations $\{(b_1,\nu_1,y_1),\ldots,(b_k,\nu_k,y_k)\}$ of the basic set $\{(a_1,\mu_1,x_1),\ldots,(a_k,\mu_k,x_k)\}$. The Fourier transform of the path-ordered tensor-function $V^{\mu_1..\mu_k}_{\mathrm{NL},\,n}(x_1,..,x_k)$ can be expressed in terms of the path-ordered tensor-functions introduced in the previous section:

$$\begin{aligned}
\tilde{V}^{\mu_1..\mu_k}_{\mathrm{NL},\,n}(q_1,..,q_k) \;=\;& \frac{1}{2}\,T^{\mu_1\mu_2}(q_1,q_2)\,\tilde{\mathcal{A}}^{\mu_3..\mu_k}_n(q_1,q_2|q_3,..,q_k)\\
+\;& \frac{1}{4}\,A^{\mu_1,\,\mu_2\mu_k}(q_1)\,\tilde{\mathcal{A}}^{\mu_3..\mu_{k-1}}_n(q_1,q_2+q_k|q_3,..,q_{k-1})\\
-\;& \frac{1}{4}\,A^{\mu_2,\,\mu_1\mu_3}(q_2)\,\tilde{\mathcal{A}}^{\mu_4..\mu_k}_n(q_1+q_3,q_2|q_4,..,q_k)\\
-\;& \frac{1}{4}\,g^{\mu_1\mu_2}\,g^{\mu_3\mu_k}\,\tilde{\mathcal{A}}^{\mu_4..\mu_{k-1}}_n(q_1+q_3,q_2+q_k|q_4,..,q_{k-1}),
\end{aligned} \tag{40}$$

where the first term contributes for $k\geq 2$, the second/third term for $k\geq 3$, and the fourth term for $k\geq 4$. Here we introduced the transverse tensors

$$T^{\mu\nu}(p,q) = (p\cdot q)\,g^{\mu\nu} - p^\nu q^\mu, \qquad A^{\mu,\,\nu\rho}(q) = g^{\mu\nu}\,q^\rho - g^{\mu\rho}\,q^\nu. \tag{41}$$

These tensors have the following properties: $p^\mu\,T^{\mu\nu}(p,q) = T^{\mu\nu}(p,q)\,q^\nu = q^\mu A^{\mu,\,\nu\rho}(q) = 0$, $p^\rho A^{\mu,\,\nu\rho}(q) = T^{\mu\nu}(q,p)$ and $p^\nu A^{\mu,\,\nu\rho}(q) = -T^{\mu\rho}(q,p)$. Using in addition the properties of the tensor-functions $\tilde{\mathcal{A}}_n$ given in App. A, one can verify that the general non-local gauge-boson interaction satisfies the (ghost-free) Ward identity

$$q_{r,\,\mu_r}\,\tilde{\Gamma}^{a_1..a_k,\,\mu_1..\mu_k}_{\mathrm{NL}}(q_1,..,q_k) = -\sum_{j\neq r}(\mathrm{F}^{a_r})^{a_j d}\left[\tilde{\Gamma}^{a_1..<a_r>..a_k,\,\mu_1..<\mu_r>..\mu_k}_{\mathrm{NL}}(q_1,..,<q_r>,..,q_k)\right]_{\substack{a_j\to d\\ q_j\to q_j+q_r}}. \tag{42}$$

On top of that, the non-local three-point interaction exhibits the proper infrared behaviour,

$$\tilde{\Gamma}^{a_1a_2a_3,\,\mu_1\mu_2\mu_3}_{\mathrm{NL}}(q_1,q_2,q_3) \xrightarrow{q_1\to 0} -(\mathrm{F}^{a_1})^{a_2a_3}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2+q_3)\,\frac{\partial}{\partial q_{2,\,\mu_1}}\Big\{T^{\mu_2\mu_3}(q_2,-q_2)\,\tilde{\Sigma}_{\mathrm{NL}}(q_2^2)\Big\}, \tag{43}$$

thereby guaranteeing the usual eikonal factorization in the infrared limit.

In addition to the non-local contributions, the three- and four-point gauge-boson interactions also receive contributions from the local action. With our conventions these local contributions read

$$ig^{k-2}\,\Gamma_{\rm L}^{a_1..a_k,\,\mu_1..\mu_k}(x_1,..,x_k) = ig^{k-2}\sum_{\rm perm}({\rm F}^{b_3}\cdots{\rm F}^{b_k})^{b_1b_2}\,V_{\rm L}^{\nu_1..\nu_k}(y_1,..,y_k), \tag{44}$$

with

$$\begin{aligned}\tilde V_{\rm L}^{\mu_1\mu_2\mu_3}(q_1,q_2,q_3) &= \frac{1}{2}\,A^{\mu_1,\,\mu_2\mu_3}(q_1)\,(2\pi)^4\,\delta^{(4)}(q_1+q_2+q_3),\\ \tilde V_{\rm L}^{\mu_1\mu_2\mu_3\mu_4}(q_1,q_2,q_3,q_4) &= -\frac{1}{4}\,g^{\mu_1\mu_2}g^{\mu_3\mu_4}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2+q_3+q_4),\\ \tilde V_{\rm L}^{\mu_1..\mu_k}(q_1,..,q_k) &= 0 \qquad (k>4).\end{aligned} \tag{45}$$

Although the above-described non-local procedure provides a gauge-invariant framework for performing the Dyson resummation of the gauge-boson self-energies, we want to stress that it is not unique. We have seen in Sect. 2 that the FLS provides a different solution of the system of gauge-boson Ward identities. In the context of non-local effective Lagrangians it is always possible to add additional towers of gauge-boson interactions that start with three-point interactions and therefore do not influence the Dyson resummation of the gauge-boson self-energies. For instance, the non-local action

$$\mathcal{S}'_{\rm NL} = g\int d^4x\,d^4y\,d^4z\,V(x,y,z)\,{\rm Tr}\Big[U(z,x)\,\boldsymbol{F}^{\mu}{}_{\nu}(x)\,U(x,y)\,\boldsymbol{F}^{\nu}{}_{\rho}(y)\,U(y,z)\,\boldsymbol{F}^{\rho}{}_{\mu}(z)\Big] \tag{46}$$

is gauge-invariant and does not affect the gauge-boson self-energies. It does contribute, however, to the interaction between three or more gauge bosons. As such it leads to a zero-mode solution of the system of gauge-boson Ward identities. In our quest for minimality we have opted to leave out such zero-mode solutions, as they are anyhow immaterial for the discussion of self-energies. In the light of the discussion presented in Sect. 2, we rearrange the series on the right-hand side of Eq. (6) according to gauge-invariant towers of gauge-boson interactions labelled by the minimum number of gauge bosons that are involved in the non-local interaction. Effectively this constitutes an expansion in powers of the coupling constant g, since a higher minimum number of particles in the interaction is equivalent to a higher minimum order in g. In order to achieve minimality we have truncated this series at the lowest effective order.

5 Unstable particles in the Standard Model

In this section we address the case of phenomenological interest: unstable particles in a broken $[SU(3)_C\times]SU(2)_L\times U(1)_Y$ gauge theory. First we briefly fix the notations. The $SU(2)_L\times U(1)_Y$ gauge-group element is defined as

$$G(x) = \exp\Big[ig_2{\rm T}^a\theta^a(x) - ig_1\frac{Y}{2}\,\theta_Y(x)\Big], \tag{47}$$

where the $SU(2)$ generators T^a can be expressed in terms of the standard Pauli spin matrices σ^a $(a = 1, 2, 3)$ according to $\mathrm{T}^a = \sigma^a/2$. The normalization condition and commutation relation read $\mathrm{Tr}\,(\sigma^a \sigma^b) = 2\delta^{ab}$ and $\left[\sigma^a, \sigma^b\right] = 2i\epsilon^{abc}\,\sigma^c$, with the $SU(2)$ structure constant ϵ^{abc} given by

$$\epsilon^{abc} = \begin{cases} +1 & \text{if } (a,b,c) = \text{ even permutation of } (1,2,3) \\ -1 & \text{if } (a,b,c) = \text{ odd permutation of } (1,2,3) \\ \ \ 0 & \text{else} \end{cases} . \tag{48}$$

The $SU(2)$ generators in the adjoint representation are given by $(\mathrm{F}^a)^{bc} = -i\epsilon^{abc}$.

5.1 The gauge bosons

In the Standard Model there are four gauge fields, W_μ^a $(a = 1, 2, 3)$ and B_μ, with the corresponding field-strength tensors given by

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu - ig_2 \left[\boldsymbol{W}_\mu, \boldsymbol{W}_\nu\right], \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{49}$$

using the shorthand notations

$$\boldsymbol{F}_{\mu\nu} \equiv \mathrm{T}^a F_{\mu\nu}^a, \qquad \boldsymbol{W}_\mu \equiv \mathrm{T}^a W_\mu^a. \tag{50}$$

In this notation the Yang–Mills Lagrangian reads:

$$\mathcal{L}^{\mathrm{YM}}(x) = -\frac{1}{2}\,\mathrm{Tr}\left[\boldsymbol{F}_{\mu\nu}(x)\,\boldsymbol{F}^{\mu\nu}(x)\right] - \frac{1}{4}\,B_{\mu\nu}(x)\,B^{\mu\nu}(x). \tag{51}$$

The physically observable gauge-boson states are given by

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i\,W_\mu^2), \quad Z_\mu = c_W W_\mu^3 + s_W B_\mu, \quad A_\mu = c_W B_\mu - s_W W_\mu^3, \tag{52}$$

where $c_W = g_2/\sqrt{g_1^2 + g_2^2}$ and $s_W = \sqrt{1 - c_W^2}$ are the cosine and sine of the weak mixing angle. The electric charge is given by $e = g_1 g_2/\sqrt{g_1^2 + g_2^2}$.

For the gauge-invariant treatment of unstable gauge bosons we can use a non-local Lagrangian that generates the relevant self-energy effects. The corresponding action can be split into two pieces. One piece is already known from the unbroken theory, bearing in mind that we have two field-strength tensors to work with:

$$\begin{aligned} \mathcal{S}_{\mathrm{NL}}^{\mathrm{YM}} &= -\frac{1}{4}\int d^4x\, d^4y\, \Sigma_1(x-y)\, B_{\mu\nu}(x)\, B^{\mu\nu}(y) \\ &\quad -\frac{1}{2}\int d^4x\, d^4y\, \Sigma_2(x-y)\, \mathrm{Tr}\left[U_2(y,x)\, \boldsymbol{F}_{\mu\nu}(x)\, U_2(x,y)\, \boldsymbol{F}^{\mu\nu}(y)\right]. \end{aligned} \tag{53}$$

Note that the path-ordered exponentials vanish in the first term. They are also not needed, since $B_{\mu\nu}(x)$ is gauge-invariant by itself. In the second term U_2 is the path-ordered exponential corresponding to $SU(2)_L$ [defined according to Eq. (9)]. Furthermore, it is impossible to

construct a gauge-invariant non-local operator of the form $B \cdot F$ using only gauge fields. Such interactions require some additional fields with non-zero vacuum expectation value, i.e. the Higgs fields. For the second piece of the non-local Lagrangian we therefore exploit the fact that the theory is spontaneously broken:

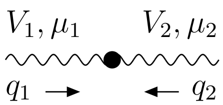
$$\begin{aligned}\mathcal{S}_{\rm NL}^{\Phi} &= -\frac{g_1 g_2}{2M_W^2}\int d^4x\, d^4y\, \Sigma_3(x-y)\,[\Phi^\dagger(x)\,\boldsymbol{F}_{\mu\nu}(x)\,\Phi(x)]\,B^{\mu\nu}(y) \\ &\quad -\frac{g_2^4}{4M_W^4}\int d^4x\, d^4y\, \Sigma_4(x-y)\,[\Phi^\dagger(x)\,\boldsymbol{F}_{\mu\nu}(x)\,\Phi(x)]\,[\Phi^\dagger(y)\,\boldsymbol{F}^{\mu\nu}(y)\,\Phi(y)], \qquad (54)\end{aligned}$$

with the $(Y=1)$ Higgs doublet Φ given by

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ [v + H(x) + i\,\chi(x)]/\sqrt{2} \end{pmatrix}. \qquad (55)$$

The non-zero vacuum expectation value v is given by $v = 2M_W/g_2$. The field operators contained in this additional effective action are clearly of higher dimension than the ones contained in the previously encountered effective actions (see the prefactors $1/M_W^2$ and $1/M_W^4$). As such these higher-dimensional operators have no local analogue in the Standard Model Lagrangian. They are required for achieving an explicit breaking of the $SU(2)$ symmetry amongst the $SU(2)$ gauge bosons in the transverse sector. After all, the loop effects in the Standard Model also lead to such explicit symmetry-breaking effects.

For completeness we now list the two-point gauge-boson interactions induced by the above-specified non-local operators:



$$: \; i\,\tilde{\Sigma}_{\rm NL}^{\mu\nu}(q_1,q_2) = i\left(q_1^\mu q_1^\nu - q_1^2 g^{\mu\nu}\right)\tilde{\Pi}_{\rm NL}^{V_1V_2}(q_1^2)\,(2\pi)^4\,\delta^{(4)}(q_1+q_2), \qquad (56)$$

with the transverse (dimensionless) self-energies given by

$$\begin{aligned}\tilde{\Pi}_{\rm NL}^{WW}(q_1^2) &= \tilde{\Sigma}_2(q_1^2) \\ \tilde{\Pi}_{\rm NL}^{ZZ}(q_1^2) &= s_W^2\,\tilde{\Sigma}_1(q_1^2) + c_W^2\,\tilde{\Sigma}_2(q_1^2) - 2s_W^2\,\tilde{\Sigma}_3(q_1^2) + c_W^2\,\tilde{\Sigma}_4(q_1^2) \\ \tilde{\Pi}_{\rm NL}^{\gamma Z}(q_1^2) &= \tilde{\Pi}_{\rm NL}^{Z\gamma}(q_1^2) \;=\; s_W c_W\,\tilde{\Sigma}_1(q_1^2) - s_W c_W\,\tilde{\Sigma}_2(q_1^2) + (s_W^2 - c_W^2)\,\frac{s_W}{c_W}\,\tilde{\Sigma}_3(q_1^2) - s_W c_W\,\tilde{\Sigma}_4(q_1^2) \\ \tilde{\Pi}_{\rm NL}^{\gamma\gamma}(q_1^2) &= c_W^2\,\tilde{\Sigma}_1(q_1^2) + s_W^2\,\tilde{\Sigma}_2(q_1^2) + 2s_W^2\,\tilde{\Sigma}_3(q_1^2) + s_W^2\,\tilde{\Sigma}_4(q_1^2). \qquad (57)\end{aligned}$$

Thus four self-energies are parametrized by four independent functions. As such all mass effects can be taken into account properly. If the theory would have been unbroken, only two functions ($\Sigma_{1,2}$) would be available for parametrizing the four self-energies in the massless limit and two relations among the self-energies would emerge. These relations hold indeed in the FLS if all fermions are massless (including the top-quark) [4]. At high energies Σ_3 and Σ_4 should vanish, since effectively the theory becomes unbroken.

At this point we remind the reader that we have only considered non-local contributions to the transverse gauge-boson self-energies, which can be resummed into dressed transverse

gauge-boson propagators. In principle one should add also non-local terms that contribute to the longitudinal gauge-boson self-energies ($\tilde{\Sigma}^{\mu\nu}_{\rm long} \propto q_1^\mu q_1^\nu$), which can be resummed into dressed longitudinal gauge-boson propagators. Since the resummation in the transverse and longitudinal sectors can be performed independently, the longitudinal sector with its close relation to the gauge-fixing procedure can be treated separately. In view of minimality we refrain from adding non-local longitudinal terms. In physical matrix elements the longitudinal propagators do not generate resonances and therefore there is no strict need for resumming (imaginary parts of) longitudinal self-energies. The imaginary parts that appear in the resummed transverse propagators of the W, Z bosons are directly linked to the corresponding decay widths $\Gamma_{W,Z}$ and are hence sufficient for a proper description of the resonance effects. In the covariant R_ξ gauge

$$\begin{aligned}
\mathcal{L}^{\rm gauge\ fix}_{R_\xi}(x) &= -\frac{1}{2}\Bigg\{\frac{1}{\xi_\gamma}\Big[\partial^\mu A_\mu(x)\Big]^2 + \frac{1}{\xi_Z}\Big[\partial^\mu Z_\mu(x) - \xi_Z M_Z \chi(x)\Big]^2 \\
&+ \frac{2}{\xi_W}\Big[\partial^\mu W^+_\mu(x) - i\xi_W M_W \phi^+(x)\Big]\Big[\partial^\nu W^-_\nu(x) + i\xi_W M_W \phi^-(x)\Big]\Bigg\}, \qquad (58)
\end{aligned}$$

for instance, we obtain the following dressed gauge-boson propagators ($V = \gamma, Z, W$):

$$\begin{aligned}
P^{VV}_{\mu\nu}(q,\xi_{_V}) &= -i\, D^{VV}_T(q^2)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - \frac{i\xi_{_V}}{q^2 - \xi_{_V} M_V^2}\,\frac{q_\mu q_\nu}{q^2}, \\
P^{\gamma Z}_{\mu\nu}(q) &= P^{Z\gamma}_{\mu\nu}(q) = -i\, D^{\gamma Z}_T(q^2)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right). \qquad (59)
\end{aligned}$$

The transverse propagator functions D_T are given by

$$\begin{aligned}
D^{WW}_T(q^2) &= \Big\{q^2 - M_W^2 + q^2\, \tilde{\Pi}^{WW}_{\rm NL}(q^2)\Big\}^{-1}, \\
D^{\gamma\gamma}_T(q^2) &= \Big[q^2 - M_Z^2 + q^2\, \tilde{\Pi}^{ZZ}_{\rm NL}(q^2)\Big]\Big/D(q^2), \\
D^{ZZ}_T(q^2) &= \Big[q^2 + q^2\, \tilde{\Pi}^{\gamma\gamma}_{\rm NL}(q^2)\Big]\Big/D(q^2), \\
D^{\gamma Z}_T(q^2) &= -q^2\, \tilde{\Pi}^{\gamma Z}_{\rm NL}(q^2)/D(q^2), \\
D(q^2) &= \Big[q^2 - M_Z^2 + q^2\, \tilde{\Pi}^{ZZ}_{\rm NL}(q^2)\Big]\Big[q^2 + q^2\, \tilde{\Pi}^{\gamma\gamma}_{\rm NL}(q^2)\Big] - \Big[q^2\, \tilde{\Pi}^{\gamma Z}_{\rm NL}(q^2)\Big]^2, \qquad (60)
\end{aligned}$$

with the explicit mass terms originating from the Higgs part of the Standard Model Lagrangian [see Eq. (62) below]. However, this is not the complete story. We will have to redefine the photon field and the electromagnetic coupling, which are by definition identified by means of the $ee\gamma$ interaction in the Thomson limit ($q_\gamma^2 = 0$). Since the $ee\gamma$ vertex does not receive non-local contributions, only the non-local photonic self-energy contributions have to be adjusted. This results in a 'finite renormalization' of the form

$$\begin{aligned}
&q^2\, D^{\gamma\gamma}_T(q^2)\Big|_{q^2=0} = \frac{1}{1 + \tilde{\Pi}^{\gamma\gamma}_{\rm NL}(0)} \to 1, \\
&q^2\, \tilde{\Pi}^{\gamma Z}_{\rm NL}(q^2)\Big|_{q^2=0} \to 0. \qquad (61)
\end{aligned}$$

In App. B we list the Feynman rules for the relevant non-local three- and four-point interactions, needed for a gauge-invariant treatment of reactions like $2f \to 4f$, $4f\gamma$, $6f$.

5.2 The Higgs boson

In the Standard Model the Higgs part of the Lagrangian is given by

$$\mathcal{L}^H(x) = \Big(D_\mu \Phi(x)\Big)^\dagger \Big(D^\mu \Phi(x)\Big) + \mu^2 \Big[\Phi^\dagger(x)\,\Phi(x)\Big] - \frac{\lambda}{4}\Big[\Phi^\dagger(x)\,\Phi(x)\Big]^2, \tag{62}$$

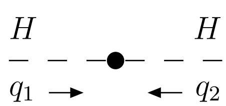
with the covariant derivative defined as

$$D_\mu = \partial_\mu - ig_2 \mathrm{T}^a W_\mu^a + ig_1 \frac{Y}{2}\, B_\mu. \tag{63}$$

The ($Y = 1$) complex Higgs doublet Φ is expanded around its vacuum expectation value $< |\phi| >_0 = v/\sqrt{2} = \sqrt{2}\,\mu/\sqrt{\lambda}$ according to Eq. (55). The resulting Lagrangian describes one physical scalar particle H with mass $M_H = \sqrt{2}\,\mu$ and three degrees of freedom that are absorbed by the gauge bosons and that are hence rendered unphysical. Our aim in this subsection is to construct an effective Lagrangian that generates a self-energy for the physical Higgs boson. At the same time we want to avoid generating any self-energies for the unphysical Higgs modes or the corresponding longitudinal gauge-boson modes. This is based on the same philosophy as adopted in the previous subsection. In order to achieve this aim we are led to a construction with only singlets, i.e. without path-ordered exponentials:

$$\mathcal{S}^H_{\mathrm{NL}} = \frac{1}{2v^2}\int d^4x\, d^4y\, \Sigma_H(x-y)\left(\Big|\Phi(x)\Big|^2 - \frac{v^2}{2}\right)\left(\Big|\Phi(y)\Big|^2 - \frac{v^2}{2}\right). \tag{64}$$

This Lagrangian induces the required self-energy in the physical Higgs propagator, without generating additional self-energy or tadpole contributions. The combined local and non-local contributions to the two-point interaction between physical Higgs bosons read:



$$: \; i\,\tilde{\Sigma}(q_1, q_2) = i\left[q_1^2 - M_H^2 + \tilde{\Sigma}_H(q_1^2)\right](2\pi)^4\,\delta^{(4)}(q_1+q_2), \tag{65}$$

which can be inverted trivially to give the dressed Higgs-boson propagator

$$P^H(q) = \frac{i}{q^2 - M_H^2 + \tilde{\Sigma}_H(q^2)}. \tag{66}$$

The remaining non-local scalar interactions can be found in App. B.

This time we can explicitly check the proposed gauge-invariant resummation procedure by considering reactions like $\phi^+\phi^- \to \chi\chi$ in the limit $M_H^2 \gg q^2 \gg M_{W,Z}^2$. Indeed, we find that in leading approximation the vertex and box graphs are identical to the non-local three- and four-point interactions, provided that tadpole renormalization is applied. This is caused by the fact that the scalar three- and four-point functions reduce to two-point functions as a result of the exchange of the heavy (physical) Higgs bosons. It should be noted that the check only works for the leading terms, since the sub-leading contributions will already contain information on higher-order non-local towers (e.g. the ones that start at three-point level).

5.3 The top-quark

In the Standard Model the fermions acquire their mass through the Yukawa interactions with the Higgs field. Since all masses are basically different, the $SU(2)$ symmetry is explicitly broken in the fermionic doublets. As such the method described in Sect. 3 is not applicable to the resummation of fermion self-energies in the Standard Model. In this subsection we concentrate on the top-quark, which has a large decay width and therefore can be described by means of perturbative methods. We start off by writing down the Yukawa interaction for the top-quark:

$$\mathcal{L}^t_{\rm Yu}(x) = -\, f_t\, \overline{Q}_L(x)\, \tilde{\Phi}(x)\, t_R(x) + \text{h.c.}, \tag{67}$$

where f_t is the top-quark Yukawa coupling. The doublet $\overline{Q}_L(x)$ and singlet $t_R(x)$ can be expressed in terms of the top-quark and bottom-quark fields $t(x), b(x)$ according to

$$Q_L(x) = \begin{pmatrix} t_L(x) \\ b_L(x) \end{pmatrix} \tag{68}$$

with

$$t_L(x) = \frac{1-\gamma_5}{2}\, t(x), \qquad t_R(x) = \frac{1+\gamma_5}{2}\, t(x) \quad \text{and} \quad b_L(x) = \frac{1-\gamma_5}{2}\, b(x). \tag{69}$$

In order to give a mass to the top-quark, a Higgs doublet $\tilde{\Phi}$ with opposite hypercharge ($Y=-1$) is required:

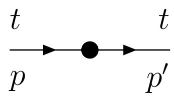
$$\tilde{\Phi}(x) = \begin{pmatrix} [v + H(x) - i\,\chi(x)]/\sqrt{2} \\ -\,\phi^-(x) \end{pmatrix} \quad \text{with} \quad \phi^-(x) = \Big[\phi^+(x)\Big]^\dagger. \tag{70}$$

The resulting top-quark mass is given by $m_t = v f_t/\sqrt{2}$. Based on this Yukawa interaction it is not difficult to construct a non-local effective action that generates a mass-like top-quark self-energy. There are two ways to non-localize the three fields in Eq. (67). The first one involves a non-local interaction between $SU(2)$ singlets:

$$\mathcal{S}^t_{\rm NL} = \frac{\sqrt{2}}{v} \int d^4x\, d^4y\, \Sigma_t(x-y) \Big\{ \Big[\, \overline{Q}_L(x)\, \tilde{\Phi}(x) \,\Big]\, U_1(x,y)\, U_3(x,y)\, t_R(y) + \text{h.c.} \Big\}, \tag{71}$$

where U_1 and U_3 are the path-ordered exponentials corresponding to $U(1)_Y$ (for $Y=4/3$) and $SU(3)_C$, respectively. The latter path-ordered exponential enters as a result of the fact that the top-quark also carries a colour charge. The second way of non-localizing Eq. (67) involves a non-local interaction between the $SU(2)$ doublets $\overline{Q}_L(x)$ and $[\tilde{\Phi}(y) t_R(y)]$, connected by a string of path-ordered exponentials $U_1(x,y)\, U_2(x,y)\, U_3(x,y)$. Note, that both effective Lagrangians contribute to the top-quark two-point interaction in the same way and therefore both allow a gauge-invariant resummation of the self-energy. A particular choice can be made on the basis of either explicit physical requirements (like the properties under parity transformations) or minimality considerations. In this paper we consider in detail the simplest of the two effective Lagrangians, given by Eq. (71).

The combined local and non-local contributions to the top-quark two-point interaction then read:



$$: \; i\,\tilde{\Sigma}(p,-p') = i\,(2\pi)^4\,\delta^{(4)}(p-p')\,\Big[\not{p} - m_t + \tilde{\Sigma}_t(p^2)\Big], \tag{72}$$

which results in the following dressed top-quark propagator:

$$P^t(p) = \frac{i}{\not{p} - m_t + \tilde{\Sigma}_t(p^2)}. \tag{73}$$

At first sight the effective action (71) seems to have little in common with the effective action introduced in Sect. 3. However, the part originating from the non-zero vacuum expectation value of the Higgs, which hence only involves fermions and gauge bosons, has the familiar form

$$\mathcal{S}^{t,v}_{\rm NL} = \int d^4x\, d^4y\, \Sigma_t(x-y)\, \bar{t}(x)\, U_1(x,y)\, U_3(x,y)\, t(y). \tag{74}$$

In App. B we list the Feynman rules for the various non-local three- and four-point interactions. Particularly noteworthy are the mixed QCD–electroweak interactions, involving both gluons and electroweak bosons, which are needed for the construction of gauge-invariant resummed amplitudes in certain mixed QCD–electroweak processes (like $e^+e^- \to t\bar{t}g$).

5.4 Some simple examples

A substantial simplification occurs when all non-local coefficients are taken to be delta-functions:

$$\Sigma_j(x-y) = \Sigma_j\,\delta(x-y) \qquad (j = 1,2,3,4,H,t). \tag{75}$$

By choosing appropriate values for the complex constants Σ_j, the simplified set of non-local actions can be used to implement the decay widths of unstable particles in a concise, gauge-invariant way. Strictly speaking, however, the proposed simplification is not supported by the actual loop effects in gauge theories, where no imaginary parts occur for space-like momenta. Nevertheless, it has become a very popular (ad hoc) procedure.

According to Eqs. (66) and (73), the simplification correponds to constant shifts in the Higgs and top-quark propagators:

$$P^H(q) = \frac{i}{q^2 - M_H^2 + \Sigma_H} \qquad \text{and} \qquad P^t(p) = \frac{i}{\not{p} - m_t + \Sigma_t}. \tag{76}$$

As a result of the delta-functions in the non-local coefficients, the effective Lagrangians defined by $\mathcal{S}^H_{\rm NL}$ in Eq. (64) and $\mathcal{S}^t_{\rm NL}$ in Eq. (71) become proportional to the corresponding local Lagrangians:

$$\mathcal{L}^H_{\rm NL}(x) = -\,\Sigma_H\,\frac{\mathcal{L}^H_{\rm pot}(x)}{M_H^2} \qquad \text{and} \qquad \mathcal{L}^t_{\rm NL}(x) = -\,\Sigma_t\,\frac{\mathcal{L}^t_{\rm Yu}(x)}{m_t}, \tag{77}$$

with $\mathcal{L}^{H}_{\text{pot}}$ indicating the part of the local Higgs Lagrangian that corresponds to the Higgs potential. From this it is clear that the combined effect of all non-local interactions amounts to the mere effective replacements

$$M_H^2 \to M_H^2 - \Sigma_H \qquad \text{and} \qquad m_t \to m_t - \Sigma_t \tag{78}$$

in the Standard Model Lagrangian. Note that for imaginary shifts (e.g. $\Sigma_H = iM_H\Gamma_H$) this procedure resembles the so-called fixed-width scheme.[1] However, in contrast to our non-local approach, the fixed-width scheme applies the effective replacements only to the propagators. So, in general the fixed-width scheme has to be adapted whenever the mass terms in the higher-point interactions play a role.

In the gauge-boson sector we need a further simplification. The higher-dimensional operators in $\mathcal{S}^{\Phi}_{\text{NL}}$ have no Standard Model analogues. Therefore, a large number of compensating higher-point interactions remain, even if all non-local coefficients are taken to be delta-functions. At this point we can exploit the fact that we don't strictly need all four non-local gauge-boson coefficients for a gauge-invariant treatment of unstable W and Z bosons. In order to properly generate the two corresponding decay widths, it is formally sufficient to have only two independent non-local coefficients. A huge simplification is achieved by setting $\Sigma_{3,4} = 0$. This comes at a price, though. As mentioned in Sect. 5.1, two relations among the gauge-boson self-energies will emerge, like in the unbroken theory. This means that the self-energies involving photons are not independent anymore. In fact this is not a real problem, since we will anyhow have to redefine the photon field and the electromagnetic coupling according to the $ee\gamma$ interaction in the Thomson limit ($q_\gamma^2 = 0$).

The net effect of the simplifications in the gauge-boson sector amounts to a rescaling of the $U(1)_Y$ and $SU(2)_L$ terms in the Yang–Mills Lagrangian (51):

$$\mathcal{L}^{\text{YM}}(x) + \mathcal{L}^{\text{YM}}_{\text{NL}}(x) = -\frac{1}{2}\,(1+\Sigma_2)\,\text{Tr}\Big[\boldsymbol{F}_{\mu\nu}(x)\,\boldsymbol{F}^{\mu\nu}(x)\Big] - \frac{1}{4}\,(1+\Sigma_1)\,B_{\mu\nu}(x)\,B^{\mu\nu}(x). \tag{79}$$

These rescaling factors can be absorbed into the gauge-boson fields and the coupling constants according to

$$W^a = \frac{W'^a}{\sqrt{1+\Sigma_2}}, \quad g_2 = g_2'\sqrt{1+\Sigma_2} \qquad \text{and} \qquad B = \frac{B'}{\sqrt{1+\Sigma_1}}, \quad g_1 = g_1'\sqrt{1+\Sigma_1}. \tag{80}$$

In terms of the redefined fields and couplings the Lagrangian (79) retrieves the original Yang–Mills form. At the same time the other Standard Model interactions are not changed by the redefinitions, as the covariant derivatives stay the same. So, the only noticeable changes involve the gauge-boson mass matrix and consequently the W^3–B mixing, which are both defined in terms of the coupling constants:

$$M_W^2 = \frac{1}{4}\,v^2 g_2^2 \quad \to \quad {M'_W}^2 = \frac{M_W^2}{1+\Sigma_2},$$

[1] By appropriately choosing the complex constants $\Sigma_{t,H}$, it is also possible to effectively replace the masses by the complex poles of the propagators [e.g. with $\Sigma_H = iM_H^2\Gamma_H/(M_H + i\,\Gamma_H)$ one obtains the effective replacement $M_H^2 \to M_H^2/(1+i\,\Gamma_H/M_H)$]. Such a complex pole mass is often better suited for the description of resonances associated with heavy unstable particles [14].

$$M_Z^2 = \frac{1}{4}v^2(g_1^2+g_2^2) \quad \to \quad {M'_Z}^2 = M_Z^2\,\frac{1+c_w^2\Sigma_1+s_w^2\Sigma_2}{(1+\Sigma_1)(1+\Sigma_2)} \equiv \frac{M_Z^2}{1+C_Z},$$
$$c_w = M_W/M_Z \quad \to \quad c'_w = M'_W/M'_Z = c_w\sqrt{\frac{1+C_Z}{1+\Sigma_2}}. \tag{81}$$

For an imaginary non-local coefficient $\Sigma_2 = i\,\Gamma_W/M_W$ the redefined W mass is identical to the so-called complex pole mass ${M'_W}^2 = (M_W^2 - iM_W\Gamma_W)/(1+\Gamma_W^2/M_W^2)$. A similar pole mass can be obtained for the Z boson by choosing Σ_1 in such a way that $C_Z = i\,\Gamma_Z/M_Z$.[2] The redefined physical states $W'^{\pm}$, Z' and A' are obtained from W'^a and B' in the usual way in terms of the redefined mixing angle. For instance, $Z' = Z\sqrt{1+C_Z}$. Since the interactions between the gauge bosons and fermions are unchanged, the A' field is by definition the photon field and the redefined coupling $e' = g'_1g'_2/\sqrt{{g'_1}^2+{g'_2}^2}$ is by definition the electromagnetic coupling constant. This is equivalent to performing finite renormalizations in order to absorb the non-local contributions to the photon wave function and the electromagnetic charge. So, the combined effect of all non-local interactions amounts to the effective replacements given in Eq. (81). These effective replacements can be extended to the longitudinal sector by simply rescaling the gauge parameter in the gauge-fixing part of the Lagrangian. For instance, with the rescaling $\xi_{W} = \xi'_{W}/(1+\Sigma_2)$ the dressed W-boson propagator in Eq. (59) becomes

$$P_{\mu\nu}^{WW}(q,\xi_W) = \frac{-i}{[1+\Sigma_2]\,[q^2-{M'_W}^2]}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right) - \frac{i\xi'_W}{[1+\Sigma_2]\,[q^2-\xi'_W{M'_W}^2]}\,\frac{q_\mu q_\nu}{q^2}, \tag{82}$$

where the factor $1/(1+\Sigma_2)$ can be absorbed into the W-boson fields according to Eq. (80). This simple example for the gauge bosons coincides with the complex-mixing-angle procedure that was adopted in Ref. [15] for calculating the radiative processes $e^+e^- \to 4f\gamma$.

The appeal of the above-discussed special examples lies in the simplicity of the net prescriptions that follow from the effective Lagrangians, allowing a straightforward implementation into the existing Monte Carlo programs. In spite of the simplicity, nevertheless a reasonably good description of the unstable-particle resonances can be achieved. In case of a more rigorous treatment of unstable particles, involving a proper energy dependence of the absorptive parts of the self-energies, one is forced to take into account the full extent of the effective Lagrangians. Or, in other words, one has to properly take into account the relevant sets of gauge-restoring multi-particle interactions (see e.g. the Feynman rules listed in App. B).

6 Conclusions and outlook

In this paper we have introduced a method that offers the possibility of performing gauge-invariant tree-level calculations with unstable particles in intermediate states. To this end non-local gauge-invariant Lagrangians are introduced, which allow the gauge-invariant resummation

[2] By choosing $\Sigma_2 = i\,\Gamma_W/(M_W - i\,\Gamma_W)$ and $C_Z = i\,\Gamma_Z/(M_Z - i\,\Gamma_Z)$ one can obtain the usual fixed-width masses ${M'_V}^2 = M_V^2 - iM_V\Gamma_V$.

of self-energies and therefore give rise to dressed (regular) propagators for unstable particles. Certainly for practical applications the resummed self-energies will be taken from the underlying gauge theory, but in principle the choice is arbitrary in our approach. For every choice the gauge restoring vertices will be different. This leaves open the possibility of studying *ad hoc* methods for implementing the decay widths of the unstable particles, like the fixed-width scheme. From the non-local Lagrangian one obtains in general an infinite set of multi-point vertices. These vertices provide an explicit solution of the full set of ghost-free Ward identities and thereby restore the gauge invariance of the resummed amplitudes. For a given multi-particle process only a limited number of those vertices contribute. In the paper we have given the derivation of the multi-point vertices from the non-local Lagrangians, and we have explicitly listed all relevant modifications of the Standard Model vertices for up to four external particles. These modified vertices are related to the unstable gauge bosons, the Higgs particle and the top-quark, which all occur in the electroweak/QCD calculations for present and future collider experiments. It should be kept in mind that there are other multi-point vertices that would also lead to gauge-invariant amplitudes. In other words, the vertices are not unique, but our prescription gives in a minimal way a set of vertices that restores gauge invariance.

Usually one restricts the final-state particles in a process to stable particles, i.e. fermions, photons and gluons. The final-state fermions can be either massive or massless. In our approach this poses no problem, since the calculation remains gauge-invariant in either case. The vertices given in App. B allow gauge-invariant calculations for unstable-particle processes like $e^+e^-/q\bar{q}/gg \to 4f\gamma$, $4fg$, $6f$. Many of the present-day unstable-particle production processes lead to these final states. Examples are $W^+W^-\gamma$ production at LEP2 and $t\bar{t}$ production at the Tevatron. For the latter process gluon radiation could also be of practical importance. In that case one should extend the list in App. B and add the vertices that contribute to a final state with six fermions and one gluon. For instance, a 3-gluon–$t\bar{t}$ vertex would arise.

Although this paper was primarily motivated by the phenomenological need to perform sensible tree-level calculations with unstable particles, other applications seem possible.

One possible application could be the gauge-invariant resummation of gluon propagators in QCD calculations. In this way part of the higher-order corrections can be taken into account in a gauge-invariant way. The effect of this resummation on multiparton amplitudes can now be investigated using our method. In a similar way one could study the resummation of the electroweak gauge-boson propagators in terms of running (effective) couplings.

Another intriguing question is whether the non-local Lagrangian technique could be used to construct a gauge-invariant bosonic self-energy by adding gauge-restoring parts from vertex and box diagrams to a non-gauge-invariant self-energy. In other words, could the method of non-local Lagrangians be used to carry out the pinch technique?

Another issue is whether one could use the propagators and vertices derived in the paper to perform quantum loop corrections. To our knowledge this remains an open question.

A Expressions for the path-ordered tensor-functions

In this appendix we derive explicit expressions for the path-ordered tensor-functions $\mathcal{A}_n$, introduced in Sect. 3. We start off the derivation by solving a set of scalar recursion relations. The simplest one is defined as

$$\mathcal{X}_n(l',l) = l^2 \mathcal{X}_{n-1}(l',l), \qquad \mathcal{X}_0(l',l) = (2\pi)^4\, \delta^{(4)}(l'+l), \tag{A.1}$$

which has the trivial solution

$$\mathcal{X}_n(l',l) = l^{2n}\, (2\pi)^4\, \delta^{(4)}(l'+l). \tag{A.2}$$

Note that such a scalar function translates directly into a non-local coefficient when the summation over n is performed: $\sum_{n=0}^{\infty} \frac{1}{n!}\, \tilde{\Sigma}_{\rm NL}^{(n)}(0)\, l^{2n} = \tilde{\Sigma}_{\rm NL}(l^2)$. For the path-ordered tensor-functions we will need to solve the following more general set of scalar recursion relations:

$$\begin{aligned} \mathcal{X}_n(l',l|q_1,..,q_k) &= l^2 \mathcal{X}_{n-1}(l',l|q_1,..,q_k) + \mathcal{X}_{n-1}(l',l+q_k|q_1,..,q_{k-1}) \qquad (k \geq 2), \\ \mathcal{X}_n(l',l|q_1) &= l^2 \mathcal{X}_{n-1}(l',l|q_1) + \mathcal{X}_{n-1}(l',l+q_1), \end{aligned} \tag{A.3}$$

with the base of the recursion given by

$$\mathcal{X}_0(l',l|q_1,..,q_k) = \mathcal{X}_0(l',l|q_1) = 0. \tag{A.4}$$

The solutions of these recursion relations read

$$\mathcal{X}_n(l',l|q_1,..,q_k) = (2\pi)^4\, \delta^{(4)}(l'+P_1) \sum_{i=1}^{k+1} \frac{P_i^{2n}}{\prod\limits_{j=1,\, j\neq i}^{k+1} \Big(P_i^2 - P_j^2\Big)}, \tag{A.5}$$

with

$$P_i = l + \sum_{j=i}^{k} q_j \qquad (i \leq k) \qquad \text{and} \qquad P_{k+1} = l. \tag{A.6}$$

Since $P_i^{2n} - l^2 P_i^{2n-2} = P_i^{2n-2}\, (P_i^2 - P_{k+1}^2)$, one can easily verify that Eq. (A.5) indeed represents a set of solutions. Note again that each term occurring in these solutions translates directly into a non-local coefficient when the summation over n is performed $[P_i^{2n} \to \tilde{\Sigma}_{\rm NL}(P_i^2)]$.

After these preparations we can now turn to the path-ordered tensor-functions

$$\tilde{\mathcal{A}}_n^{\mu_1..\mu_k}(l',l|q_1,..,q_k) = (-i)^k \int d^4\xi\, d^4\tau\, \delta^{(4)}(\xi-\tau)\, e^{-il'\cdot\xi} (-\partial_\tau^2)^n\, e^{-il\cdot\tau} \prod_{j=1}^{k} \int\limits_{\omega_{j-1}}^{\tau} d\omega_j^{\mu_j}\, e^{-iq_j\cdot\omega_j}, \tag{A.7}$$

with $\omega_0 = \xi$. By working out one of the $(-\partial_\tau^2)$ operators one arrives at the following set of tensor recursion relations:

$$\tilde{\mathcal{A}}_n(l',l) = l^2 \tilde{\mathcal{A}}_{n-1}(l',l), \tag{A.8}$$

$$
\begin{aligned}
\tilde{\mathcal{A}}_n^{\mu_1}(l',l|q_1) &= l^2\tilde{\mathcal{A}}_{n-1}^{\mu_1}(l',l|q_1) + (2l+q_1)^{\mu_1}\,\tilde{\mathcal{A}}_{n-1}(l',l+q_1),\\
\tilde{\mathcal{A}}_n^{\mu_1\mu_2}(l',l|q_1,q_2) &= l^2\tilde{\mathcal{A}}_{n-1}^{\mu_1\mu_2}(l',l|q_1,q_2) + (2l+q_2)^{\mu_2}\,\tilde{\mathcal{A}}_{n-1}^{\mu_1}(l',l+q_2|q_1)\\
&\quad + g^{\mu_1\mu_2}\tilde{\mathcal{A}}_{n-1}(l',l+q_1+q_2),\\
\tilde{\mathcal{A}}_n^{\mu_1..\mu_k}(l',l|q_1,..,q_k) &= l^2\tilde{\mathcal{A}}_{n-1}^{\mu_1..\mu_k}(l',l|q_1,..,q_k) + (2l+q_k)^{\mu_k}\,\tilde{\mathcal{A}}_{n-1}^{\mu_1..\mu_{k-1}}(l',l+q_k|q_1,..,q_{k-1})\\
&\quad + g^{\mu_{k-1}\mu_k}\tilde{\mathcal{A}}_{n-1}^{\mu_1..\mu_{k-2}}(l',l+q_{k-1}+q_k|q_1,..,q_{k-2}) \qquad (k\geq 3).
\end{aligned}
$$

The base of the recursion is given by the relations

$$
\tilde{\mathcal{A}}_0^{\mu_1..\mu_k}(l',l|q_1,..,q_k) = 0 \qquad (k\geq 1) \qquad \text{and} \qquad \tilde{\mathcal{A}}_0(l',l) = (2\pi)^4\,\delta^{(4)}(l'+l). \tag{A.9}
$$

Evidently $\tilde{\mathcal{A}}_n(l',l)$ is identical to $\mathcal{X}_n(l',l)$. The other tensor-functions can also be expressed in a straightforward way in terms of the afore-mentioned solutions of the scalar recursion relations:

$$
\begin{aligned}
\tilde{\mathcal{A}}_n^{\mu_1..\mu_k}(l',l|q_1,..,q_k) &= \mathcal{X}_n(l',l|q_1,..,q_k)\,Q_1^{\mu_1}\cdots Q_k^{\mu_k}\\
&+ \sum_{m=1}^{k-1}\mathcal{X}_n(l',l|q_1,..,q_{m-1},q_m+q_{m+1},q_{m+2},..,q_k)\,Q_1^{\mu_1}\cdots Q_{m-1}^{\mu_{m-1}}\,g^{\mu_m\mu_{m+1}}\,Q_{m+2}^{\mu_{m+2}}\cdots Q_k^{\mu_k}\\
&+ \text{ two insertions of the metric tensor } g \;+\; \cdots,
\end{aligned} \tag{A.10}
$$

with

$$
Q_i^{\mu_i} = P_i^{\mu_i} + P_{i+1}^{\mu_i}. \tag{A.11}
$$

As a final step we insert the explicit solutions (A.5):

$$
\begin{aligned}
\tilde{\mathcal{A}}_n^{\mu_1..\mu_k}(l',l|q_1,..,q_k) &= (2\pi)^4\,\delta^{(4)}(l'+P_1)\sum_{i=1}^{k+1}\frac{P_i^{2n}}{\prod\limits_{j=1,\,j\neq i}^{k+1}\Big(P_i^2-P_j^2\Big)}\;O_i^{\mu_1..\mu_k}(l|q_1,..,q_k),\\
O_i^{\mu_1..\mu_k}(l|q_1,..,q_k) &= Q_1^{\mu_1}\cdots Q_k^{\mu_k} + \sum_{m=1}^{k-1}\Big(P_i^2-P_{m+1}^2\Big)\,Q_1^{\mu_1}\cdots Q_{m-1}^{\mu_{m-1}}\,g^{\mu_m\mu_{m+1}}\,Q_{m+2}^{\mu_{m+2}}\cdots Q_k^{\mu_k}\\
&\quad + \text{ two insertions of the metric tensor } g \;+\; \cdots
\end{aligned} \tag{A.12}
$$

As a result of the relation $q_{i,\mu_i}\,Q_i^{\mu_i} = (P_i^2 - P_{i+1}^2)$, the tensor-functions obey the simple 'Ward identities'

$$
\begin{aligned}
q_{1,\mu_1}\,\tilde{\mathcal{A}}_n^{\mu_1}(l',l|q_1) &= \tilde{\mathcal{A}}_n(l',l+q_1) - \tilde{\mathcal{A}}_n(l'+q_1,l), \qquad\qquad (A.13)\\
q_{1,\mu_1}\,\tilde{\mathcal{A}}_n^{\mu_1..\mu_k}(l',l|q_1,..,q_k) &= \tilde{\mathcal{A}}_n^{\mu_2..\mu_k}(l',l|q_1+q_2,q_3,..,q_k) - \tilde{\mathcal{A}}_n^{\mu_2..\mu_k}(l'+q_1,l|q_2,..,q_k),\\
q_{k,\mu_k}\,\tilde{\mathcal{A}}_n^{\mu_1..\mu_k}(l',l|q_1,..,q_k) &= \tilde{\mathcal{A}}_n^{\mu_1..\mu_{k-1}}(l',l+q_k|q_1,..,q_{k-1}) - \tilde{\mathcal{A}}_n^{\mu_1..\mu_{k-1}}(l',l|q_1,..,q_{k-2},q_{k-1}+q_k),\\
q_{r,\mu_r}\,\tilde{\mathcal{A}}_n^{\mu_1..\mu_k}(l',l|q_1,..,q_k) &= \tilde{\mathcal{A}}_n^{\mu_1..\mu_{r-1}\mu_{r+1}..\mu_k}(l',l|q_1,..,q_{r-1},q_r+q_{r+1},q_{r+2},..,q_k)\\
&\quad - \tilde{\mathcal{A}}_n^{\mu_1..\mu_{r-1}\mu_{r+1}..\mu_k}(l',l|q_1,..,q_{r-2},q_{r-1}+q_r,q_{r+1},..,q_k) \qquad (1<r<k).
\end{aligned}
$$

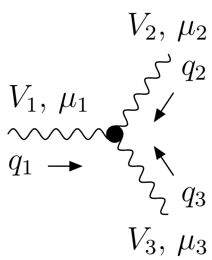
As an example we present the explicit solutions for $k = 0, 1, 2$, which are relevant for the derivation of the two-, three- and four-point interactions that are presented in this paper:

$$
\begin{aligned}
\tilde{\mathcal{A}}_n(l', l) &= (2\pi)^4\, \delta^{(4)}(l' + l)\, l^{2n}, \qquad \text{(A.14)} \\
\tilde{\mathcal{A}}_n^{\mu_1}(l', l|q_1) &= (2\pi)^4\, \delta^{(4)}(l' + l + q_1)\, \frac{(l+q_1)^{2n} - l^{2n}}{(l+q_1)^2 - l^2}\, (2l + q_1)^{\mu_1}, \\
\tilde{\mathcal{A}}_n^{\mu_1\mu_2}(l', l|q_1, q_2) &= (2\pi)^4\, \delta^{(4)}(l' + l + q_1 + q_2) \Bigg[g^{\mu_1\mu_2}\, \frac{(l+q_1+q_2)^{2n} - l^{2n}}{(l+q_1+q_2)^2 - l^2} \\
&\quad + (2l + 2q_2 + q_1)^{\mu_1}\, (2l + q_2)^{\mu_2} \bigg\{ \frac{(l+q_1+q_2)^{2n}}{[(l+q_1+q_2)^2 - (l+q_2)^2][(l+q_1+q_2)^2 - l^2]} \\
&\quad - \frac{(l+q_2)^{2n}}{[(l+q_1+q_2)^2 - (l+q_2)^2][(l+q_2)^2 - l^2]} + \frac{l^{2n}}{[(l+q_1+q_2)^2 - l^2][(l+q_2)^2 - l^2]} \bigg\} \Bigg].
\end{aligned}
$$

B Some non-local Feynman rules

In this appendix we list the non-local contributions to the various three- and four-point interactions. Whenever possible we will suppress the factor $(2\pi)^4$ and the delta-function for momentum conservation.

First we give the non-local contributions to the pure gauge-boson interactions as originating from the non-local actions $\mathcal{S}_{\rm NL}^{\rm YM}$ in Eq. (53) and $\mathcal{S}_{\rm NL}^{\Phi}$ in Eq. (54). We start with the three-point gauge-boson interaction:



$$
: \quad ig_2 \Bigg\{ A_2 \sum_{n=0}^{\infty} \frac{1}{n!}\, \tilde{\Sigma}_2^{(n)}(0) \sum_{\rm perm} \epsilon^{jkl}\, \tilde{V}_{{\rm NL},\,n}^{\mu_j\mu_k\mu_l}(q_j, q_k, q_l) + A^{\mu_1,\,\mu_2\mu_3}(q_1) \Big[A_{31}\, \tilde{\Sigma}_3(q_1^2) + A_{41}\, \tilde{\Sigma}_4(q_1^2) \Big] \Bigg\}, \qquad \text{(B.1)}
$$

with the various couplings given by

$V_1V_2V_3$	A_2	A_{31}	A_{41}
ZW^+W^-	$-c_w$	s_w^2/c_w	$-c_w$
γW^+W^-	s_w	s_w	s_w

(B.2)

The sum over the permutations involves all permutations (j, k, l) of the labels $(1, 2, 3)$. Now we can make use of Eqs. (40) and (A.14) to arrive at

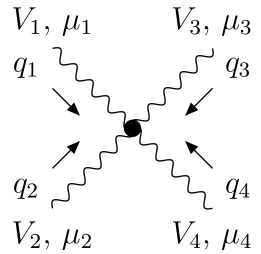
$$
\sum_{n=0}^{\infty} \frac{1}{n!}\, \tilde{\Sigma}_2^{(n)}(0)\, \tilde{V}_{{\rm NL},\,n}^{\mu_1\mu_2\mu_3}(q_1, q_2, q_3) = \sum_{n=0}^{\infty} \frac{1}{n!}\, \tilde{\Sigma}_2^{(n)}(0) \Big\{ \frac{1}{2}\, T^{\mu_1\mu_2}(q_1, q_2)\, \tilde{\mathcal{A}}_n^{\mu_3}(q_1, q_2|q_3)
$$

$$
+\,\frac{1}{4}\,A^{\mu_1,\,\mu_2\mu_3}(q_1)\,\tilde{\mathcal{A}}_n(q_1,q_2+q_3)-\frac{1}{4}\,A^{\mu_2,\,\mu_1\mu_3}(q_2)\,\tilde{\mathcal{A}}_n(q_1+q_3,q_2)\Big\}
$$

$$
\rightarrow\quad \tilde{\Sigma}_2(q_1^2)\left\{\frac{1}{2}\,A^{\mu_1,\,\mu_2\mu_3}(q_1)+\frac{(2q_1+q_3)^{\mu_3}}{(q_1+q_3)^2-q_1^2}\,T^{\mu_1\mu_2}(q_1,q_2)\right\}. \tag{B.3}
$$

In the last step we have compactified the expression by exploiting the fact that the summation over all permutations has to be performed and that ϵ^{jkl} is totally antisymmetric. Moreover, the factor $(2\pi)^4\,\delta^{(4)}(q_1+q_2+q_3)$ has been suppressed.

The four-point gauge-boson interaction is modified according to



$$
:\ \ ig_2^2\Big\{B_2\sum_{n=0}^{\infty}\frac{1}{n!}\,\tilde{\Sigma}_2^{(n)}(0)\sum_{\text{perm}}\eta_{jklm}\,\tilde{V}^{\mu_j\mu_k\mu_l\mu_m}_{\mathrm{NL},\,n}(q_j,q_k,q_l,q_m)
$$

$$
+\ B_{413}\,\tilde{\Sigma}_4([q_1+q_3]^2)\left(g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right)
$$

$$
+\ B_{414}\,\tilde{\Sigma}_4([q_1+q_4]^2)\left(g^{\mu_1\mu_2}g^{\mu_3\mu_4}-g^{\mu_1\mu_3}g^{\mu_2\mu_4}\right)\Big\}, \tag{B.4}
$$

with the various couplings given by

$V_1V_2V_3V_4$	B_2	B_{413}	B_{414}
W^+W^-ZZ	$-c_w^2$	0	0
$W^+W^-Z\gamma$	s_wc_w	0	0
$W^+W^-\gamma\gamma$	$-s_w^2$	0	0
$W^+W^+W^-W^-$	1	1	1

(B.5)

The sum over the permutations involves all permutations (j,k,l,m) of the labels $(1,2,3,4)$ and

$$
\eta_{jklm}=\begin{cases}\ \ 0 & \text{if}\ \ (j,k,l,m)\ =(1,3,2,4)\,,\ (4,2,3,1)\ \ \text{or any}\ \ 1\leftrightarrow 2\,,\ 3\leftrightarrow 4\ \ \text{permutation}\\ +1 & \text{if}\ \ (j,k,l,m)\ =(1,3,4,2)\,,\ (4,2,1,3)\ \ \text{or any}\ \ 1\leftrightarrow 2\,,\ 3\leftrightarrow 4\ \ \text{permutation}\\ -1 & \text{if}\ \ (j,k,l,m)\ =(1,2,3,4)\,,\ (3,4,1,2)\ \ \text{or any}\ \ 1\leftrightarrow 2\,,\ 3\leftrightarrow 4\ \ \text{permutation}\end{cases}. \tag{B.6}
$$

Now we can make use of Eqs. (40) and (A.14) to arrive at

$$
\sum_{n=0}^{\infty}\frac{1}{n!}\,\tilde{\Sigma}_2^{(n)}(0)\,\tilde{V}^{\mu_1\mu_2\mu_3\mu_4}_{\mathrm{NL},\,n}(q_1,q_2,q_3,q_4)=\sum_{n=0}^{\infty}\frac{1}{n!}\,\tilde{\Sigma}_2^{(n)}(0)\,\Big\{\frac{1}{2}\,T^{\mu_1\mu_2}(q_1,q_2)\,\tilde{\mathcal{A}}_n^{\mu_3\mu_4}(q_1,q_2|q_3,q_4)
$$

$$
+\,\frac{1}{4}\,A^{\mu_1,\,\mu_2\mu_4}(q_1)\,\tilde{\mathcal{A}}_n^{\mu_3}(q_1,q_2+q_4|q_3)-\frac{1}{4}\,A^{\mu_2,\,\mu_1\mu_3}(q_2)\,\tilde{\mathcal{A}}_n^{\mu_4}(q_1+q_3,q_2|q_4)
$$

$$
-\,\frac{1}{4}\,g^{\mu_1\mu_2}\,g^{\mu_3\mu_4}\,\tilde{\mathcal{A}}_n(q_1+q_3,q_2+q_4)\Big\}
$$

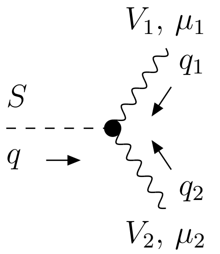
$$
\rightarrow\quad \tilde{\Sigma}_2(q_1^2)\,\frac{T^{\mu_1\mu_2}(q_1,q_2)}{q_1^2-q_2^2}\left\{g^{\mu_3\mu_4}+\frac{(2q_2+q_4)^{\mu_4}(2q_1+q_3)^{\mu_3}}{(q_1+q_3)^2-q_1^2}\right\}
$$

$$
-\frac{1}{4}\,\tilde{\Sigma}_2([q_1+q_3]^2)\left\{g^{\mu_1\mu_2}g^{\mu_3\mu_4}+2\,\frac{(2q_1+q_3)^{\mu_3}}{(q_1+q_3)^2-q_1^2}\,\frac{(2q_2+q_4)^{\mu_4}}{(q_2+q_4)^2-q_2^2}\,T^{\mu_1\mu_2}(q_1,q_2)\right\}
$$

$$
+\frac{1}{2}\,\frac{(2q_1+q_3)^{\mu_3}}{(q_1+q_3)^2-q_1^2}\,A^{\mu_1,\,\mu_2\mu_4}(q_1)\left\{\tilde{\Sigma}_2(q_1^2)-\tilde{\Sigma}_2([q_1+q_3]^2)\right\}. \tag{B.7}
$$

In the last step we have again exploited the symmetry properties of the summation over all permutations.

The non-local action $\mathcal{S}_{\rm NL}^{\Phi}$ in Eq. (54) also contains explicit interactions between gauge bosons and physical/unphysical Higgs bosons. The contribution to the interaction between one scalar particle and two gauge bosons reads



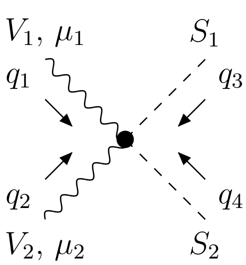
$$
:\quad \frac{ig_2}{M_W}\,T^{\mu_1\mu_2}(q_1,q_2)\left\{\frac{s_w}{c_w}\left[C_{31}\,\tilde{\Sigma}_3(q_1^2)+C_{32}\,\tilde{\Sigma}_3(q_2^2)\right]+C_{41}\,\tilde{\Sigma}_4(q_1^2)+C_{42}\,\tilde{\Sigma}_4(q_2^2)\right\}, \tag{B.8}
$$

with the various couplings given by

SV_1V_2	C_{31}	C_{32}	C_{41}	C_{42}
HZZ	$-s_wc_w$	$-s_wc_w$	c_w^2	c_w^2
$HZ\gamma$	s_w^2	$-c_w^2$	$-s_wc_w$	$-s_wc_w$
$H\gamma\gamma$	s_wc_w	s_wc_w	s_w^2	s_w^2
$\phi^{\mp}ZW^{\pm}$	s_w	0	$-c_w$	0
$\phi^{\mp}\gamma W^{\pm}$	c_w	0	s_w	0

(B.9)

For the interaction between two scalar particles and two gauge bosons we obtain



$$
:\quad \frac{ig_2^2}{2M_W^2}\,T^{\mu_1\mu_2}(q_1,q_2)\left\{\frac{s_w}{c_w}\left[D_{31}\,\tilde{\Sigma}_3(q_1^2)+D_{32}\,\tilde{\Sigma}_3(q_2^2)\right]+D_{41}\,\tilde{\Sigma}_4(q_1^2)\right.
$$

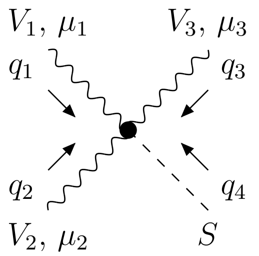
$$
\left.+D_{42}\,\tilde{\Sigma}_4(q_2^2)+2\,D_{413}\,\tilde{\Sigma}_4([q_1+q_3]^2)+2\,D_{414}\,\tilde{\Sigma}_4([q_1+q_4]^2)\right\}, \tag{B.10}
$$

with the various couplings given by

$V_1V_2S_1S_2$	D_{31}	D_{32}	D_{41}	D_{42}	D_{413}	D_{414}
$ZZHH$	$-s_wc_w$	$-s_wc_w$	c_w^2	c_w^2	c_w^2	c_w^2
$Z\gamma HH$	s_w^2	$-c_w^2$	$-s_wc_w$	$-s_wc_w$	$-s_wc_w$	$-s_wc_w$
$\gamma\gamma HH$	s_wc_w	s_wc_w	s_w^2	s_w^2	s_w^2	s_w^2
$ZZ\chi\chi$	$-s_wc_w$	$-s_wc_w$	c_w^2	c_w^2	0	0
$Z\gamma\chi\chi$	s_w^2	$-c_w^2$	$-s_wc_w$	$-s_wc_w$	0	0
$\gamma\gamma\chi\chi$	s_wc_w	s_wc_w	s_w^2	s_w^2	0	0
$ZZ\phi^+\phi^-$	s_wc_w	s_wc_w	$-c_w^2$	$-c_w^2$	0	0
$Z\gamma\phi^+\phi^-$	$-s_w^2$	c_w^2	s_wc_w	s_wc_w	0	0
$\gamma\gamma\phi^+\phi^-$	$-s_wc_w$	$-s_wc_w$	$-s_w^2$	$-s_w^2$	0	0
$ZW^\pm\phi^\mp H$	s_w	0	$-c_w$	0	0	$-c_w$
$\gamma W^\pm\phi^\mp H$	c_w	0	s_w	0	0	s_w
$ZW^\pm\phi^\mp\chi$	$\pm is_w$	0	$\mp ic_w$	0	0	0
$\gamma W^\pm\phi^\mp\chi$	$\pm ic_w$	0	$\pm is_w$	0	0	0
$W^+W^-\phi^+\phi^-$	0	0	0	0	0	1
$W^\pm W^\pm\phi^\mp\phi^\mp$	0	0	0	0	1	1

(B.11)

In addition new interactions emerge between one scalar particle and three gauge bosons:



$$: \quad \frac{ig_2^2}{M_W}\Bigg\{A^{\mu_1,\,\mu_2\mu_3}(q_1)\left[\frac{s_w}{c_w}\,E_{31}\,\tilde{\Sigma}_3(q_1^2)+E_{41}\,\tilde{\Sigma}_4(q_1^2)+E_{414}\,\tilde{\Sigma}_4([q_1+q_4]^2)\right]$$

$$+\ A^{\mu_2,\,\mu_1\mu_3}(q_2)\left[\frac{s_w}{c_w}\,E_{32}\,\tilde{\Sigma}_3(q_2^2)+E_{42}\,\tilde{\Sigma}_4(q_2^2)+E_{413}\,\tilde{\Sigma}_4([q_1+q_3]^2)\right]\Bigg\}, \tag{B.12}$$

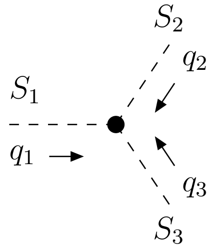
with the various couplings given by

$V_1V_2V_3S$	E_{31}	E_{32}	E_{41}	E_{42}	E_{413}	E_{414}
$ZZW^\pm\phi^\mp$	$\mp s_wc_w$	$\mp s_wc_w$	$\pm c_w^2$	$\pm c_w^2$	0	0
$Z\gamma W^\pm\phi^\mp$	$\pm s_w^2$	$\mp c_w^2$	$\mp s_wc_w$	$\mp s_wc_w$	0	0
$\gamma\gamma W^\pm\phi^\mp$	$\pm s_wc_w$	$\pm s_wc_w$	$\pm s_w^2$	$\pm s_w^2$	0	0
ZW^+W^-H	s_w	0	$-c_w$	0	0	$-c_w$
γW^+W^-H	c_w	0	s_w	0	0	s_w
$W^\pm W^\pm W^\mp\phi^\mp$	0	0	0	0	± 1	± 1

(B.13)

The non-local action $\mathcal{S}_{\rm NL}^H$ in Eq. (64) modifies the three- and four-point interactions between the physical and unphysical Higgs bosons. The contribution to the scalar three-point interaction

is



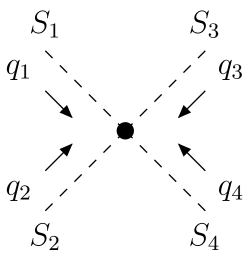
$$: \ \frac{i}{v}\left[C_1^{(3S)}\,\tilde{\Sigma}_H(q_1^2) + C_2^{(3S)}\,\tilde{\Sigma}_H(q_2^2) + C_3^{(3S)}\,\tilde{\Sigma}_H(q_3^2)\right], \tag{B.14}$$

with the various couplings given by

$S_1S_2S_3$	$C_1^{(3S)}$	$C_2^{(3S)}$	$C_3^{(3S)}$
HHH	1	1	1
$H\chi\chi$	1	0	0
$H\phi^+\phi^-$	1	0	0

(B.15)

The contribution to the scalar four-point interaction is



$$: \ \frac{i}{v^2}\left[C_{12}^{(4S)}\,\tilde{\Sigma}_H([q_1{+}q_2]^2){+}C_{13}^{(4S)}\,\tilde{\Sigma}_H([q_1{+}q_3]^2){+}C_{14}^{(4S)}\,\tilde{\Sigma}_H([q_1{+}q_4]^2)\right], \tag{B.16}$$

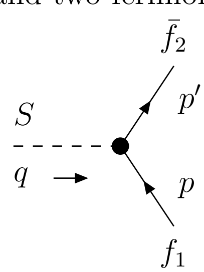
with the various couplings given by

$S_1S_2S_3S_4$	$C_{12}^{(4S)}$	$C_{13}^{(4S)}$	$C_{14}^{(4S)}$
$HHHH$	1	1	1
$\chi\chi\chi\chi$	1	1	1
$HH\chi\chi$	1	0	0
$HH\phi^+\phi^-$	1	0	0
$\chi\chi\phi^+\phi^-$	1	0	0
$\phi^+\phi^+\phi^-\phi^-$	0	1	1

(B.17)

The local scalar three- and four-point interactions can be obtained by simply replacing $\tilde{\Sigma}_H(q^2)$ by $-M_H^2$.

The non-local action $\mathcal{S}_{\rm NL}^t$ in Eq. (71), finally, modifies various three- and four-point interactions between fermions and bosons. We start with the contribution to the interaction between one scalar particle and two fermions:

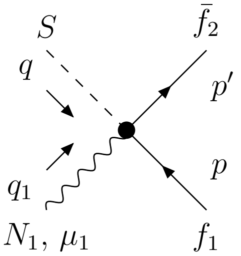


$$: \ \frac{i}{v}\left[C_+\,\omega_+\,\tilde{\Sigma}_t(p^2) + C_-\,\omega_-\,\tilde{\Sigma}_t(p'^2)\right], \tag{B.18}$$

with $\omega_\pm = (1 \pm \gamma_5)/2$. The various couplings are given by

$$
\begin{array}{|l|c|c|}
\hline
S f_1 \bar{f}_2 & C_+ & C_- \\
\hline
H t\bar{t} & 1 & 1 \\
\chi t\bar{t} & -i & i \\
\phi^- t\bar{b} & -\sqrt{2} & 0 \\
\phi^+ b\bar{t} & 0 & -\sqrt{2} \\
\hline
\end{array}
\qquad \text{(B.19)}
$$

The local top-quark Yukawa interactions can be obtained by simply replacing $\tilde{\Sigma}_t(q^2)$ by $-m_t$. Owing to the path-ordered exponentials $U_{1,3}$, the above interaction can be extended by attaching an additional neutral gauge boson N_1:

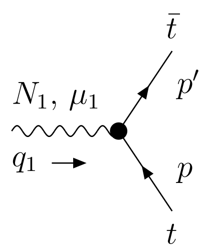


$$
\begin{aligned}
&: \ \frac{i}{v}\,\mathcal{G}(N_1) \sum_{n=0}^{\infty} \frac{1}{n!}\, \tilde{\Sigma}_t^{(n)}(0) \Big[C_+\,\omega_+\,\tilde{\mathcal{A}}_n^{\mu_1}(q-p', p|q_1) + C_-\,\omega_-\,\tilde{\mathcal{A}}_n^{\mu_1}(-p', q+p|q_1) \Big] \\
&\to \frac{i}{v}\,\mathcal{G}(N_1) \Big[C_+\,\omega_+\,(p+p'-q)^{\mu_1}\, \frac{\tilde{\Sigma}_t([q-p']^2) - \tilde{\Sigma}_t(p^2)}{(q-p')^2 - p^2} \\
&\qquad + \ C_-\,\omega_-\,(p+p'+q)^{\mu_1}\, \frac{\tilde{\Sigma}_t(p'^2) - \tilde{\Sigma}_t([p+q]^2)}{p'^2 - (p+q)^2} \Big]. \qquad \text{(B.20)}
\end{aligned}
$$

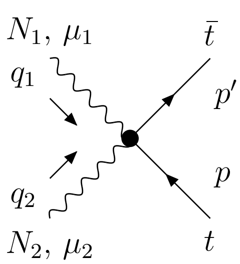
The couplings $C_\pm$ are the same as without the neutral gauge boson and the generalized gauge coupling $\mathcal{G}(N_1)$ is defined as

$$
\mathcal{G}(N_j) = \{-Q_t\,e; -Q_t\,e\,s_w/c_w; g_s \mathrm{T}^{a_j}\} \qquad \text{for} \qquad N_j = \{\gamma; Z; g^{a_j}\}, \qquad \text{(B.21)}
$$

with $Q_t = 2/3$ denoting the charge of the top-quark in units of e. The part of $\mathcal{S}_{\mathrm{NL}}^t$ that originates from the non-zero vacuum expectation value of the Higgs, $\mathcal{S}_{\mathrm{NL}}^{t,v}$, involves fermions and gauge bosons only [see Eq. (74)]. The corresponding Feynman rules resemble the ones derived in Sect. 3. For the three- and four-point interactions we find



$$
\begin{aligned}
&: \ i\,\mathcal{G}(N_1) \sum_{n=0}^{\infty} \frac{1}{n!}\, \tilde{\Sigma}_t^{(n)}(0)\, \tilde{\mathcal{A}}_n^{\mu_1}(-p', p|q_1) \\
&\to \ i\,\mathcal{G}(N_1)\,(p+p')^{\mu_1}\, \frac{\tilde{\Sigma}_t(p'^2) - \tilde{\Sigma}_t(p^2)}{p'^2 - p^2}, \qquad \text{(B.22)}
\end{aligned}
$$



$$
: \ i \sum_{n=0}^{\infty} \frac{1}{n!}\, \tilde{\Sigma}_t^{(n)}(0) \Big[\mathcal{G}(N_1)\,\mathcal{G}(N_2)\,\tilde{\mathcal{A}}_n^{\mu_1\mu_2}(-p', p|q_1, q_2) + (1 \leftrightarrow 2) \Big]
$$

$$
\begin{aligned}
\to\quad & i\,\frac{\mathcal{G}(N_1)\,\mathcal{G}(N_2)}{p'^{\,2}-p^2}\left[\frac{(p'+p+q_2)^{\mu_1}}{p'^{\,2}-(p+q_2)^2}\,\frac{(2p+q_2)^{\mu_2}}{p^2-(p+q_2)^2}\left\{\left[p^2-(p+q_2)^2\right]\tilde{\Sigma}_t(p'^{\,2})\right.\right.\\
&\qquad\left. +\left[p'^{\,2}-p^2\right]\tilde{\Sigma}_t([p+q_2]^2)+\left[(p+q_2)^2-p'^{\,2}\right]\tilde{\Sigma}_t(p^2)\right\}\\
&\qquad\left. +\,g^{\mu_1\mu_2}\left\{\tilde{\Sigma}_t(p'^{\,2})-\tilde{\Sigma}_t(p^2)\right\}\right]\\
& +(N_1,\mu_1,q_1)\leftrightarrow(N_2,\mu_2,q_2).
\end{aligned}
\tag{B.23}
$$

References

[1] F. A. Berends and G. B. West, *Phys. Rev.* **D1** (1970) 122.

[2] Y. Kurihara, D. Perret-Gallix and Y. Shimizu, *Phys. Lett.* **B349** (1995) 367.

[3] E.N. Argyres *et al.*, *Phys. Lett.* **B358** (1995) 339.

[4] W. Beenakker *et al.*, *Nucl. Phys.* **B500** (1997) 255.

[5] M. Veltman, *Physica* **29** (1963) 186;
R.G. Stuart, *Phys. Lett.* **B262** (1991) 113;
A. Aeppli, G.J. van Oldenborgh and D. Wyler, *Nucl. Phys.* **B428** (1994) 126.

[6] W. Beenakker, F.A. Berends and A.P. Chapovsky, *Nucl. Phys.* **B548** (1999) 3.

[7] U. Baur and D. Zeppenfeld, *Phys. Rev. Lett.* **75** (1995) 1002.

[8] M. Beuthe, R. Gonzalez Felipe, G. Lopez Castro and J. Pestieau, *Nucl. Phys.* **B498** (1997) 55;
G. Passarino, *hep–ph/*9911482;
E. Accomando, A. Ballestrero, E. Maina, *hep–ph/*9911489.

[9] J.M. Cornwall, *Phys. Rev.* **D26** (1982) 1453;
J.M. Cornwall and J. Papavassiliou, *Phys. Rev.* **D40** (1989) 3474;
J. Papavassiliou, *Phys. Rev.* **D41** (1990) 3179;
G. Degrassi and A. Sirlin, *Phys. Rev.* **D46** (1992) 3104;
N.J. Watson, *Nucl. Phys.* **B494** (1997) 388.

[10] J. Papavassiliou and A. Pilaftsis, *Phys. Rev.* **D53** (1996) 2128; *Phys. Rev.* **D54** (1996) 5315.

[11] A. Denner and S. Dittmaier, *Phys. Rev.* **D54** (1996) 4499.

[12] N.J. Watson, *Nucl. Phys.* **B552** (1999) 461.

[13] J. Terning, *Phys. Rev.* **D44** (1991) 887.

[14] G. Lopez Castro, J.L. Lucio and J. Pestieau, *Mod. Phys. Lett.* **A6** (1991) 3679;
S. Willenbrock and G. Valencia, *Phys. Lett.* **B259** (1991) 373;
W. Beenakker, J. Hoogland, R. Kleiss and G.J. van Oldenborgh, *Phys. Lett.* **B376** (1996) 136.

[15] A. Denner, S. Dittmaier, M. Roth and D. Wackeroth, *hep–ph/*9904472.